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# Multipole Moments in Stationary Spacetimes

Master Thesis in Mathematics
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**Abstract**

Multipole moments are a powerful tool in general relativity. They provide a detailed description of the geometry of the spacetime, encoding information about mass and angular momentum. The lowest order multipole moments can also be measured for some objects, allowing to compare the theory with experiments. However, defining multipole moments in general relativity is challenging. This thesis restricts to stationary asymptotically flat spacetimes, starting with an extensive discussion of these key assumptions. A critical aspect is ensuring the uniqueness in Geroch's definition of asymptotic flatness [42]. We identify issues with Geroch's original result and propose a corrected version.

In stationary asymptotically flat spacetimes, several exact definitions of multipole moments exist. Most notably, there are the Geroch–Hansen [48] and Thorne [107] formalisms in vacuum, which are explored in this thesis. Despite their very different definitions, the multipole moments due to Geroch–Hansen and Thorne are equivalent.

Moving beyond vacuum, we investigate multipole moments in electrovacuum. There exists a natural extension of the Geroch–Hansen formalism to electrovacuum and we propose an extension of the Thorne formalism. Recent work by Mayerson [77] defines gravitational multipole moments in rather general non-vacuum solutions of the Einstein equations, but these alone do not sufficiently distinguish spacetimes. To see multipole moments at work in another specific class of non-vacuum solutions where we can define natural matter multipole moments, we extend the Geroch–Hansen formalism to scalar field solutions of the Einstein equations.

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# Contents

# Chapter 1

# Introduction

In electromagnetism, a multipole expansion is a very powerful technique to study sources that are contained in some bounded region. In that case, we can expand the electric and magnetic potentials in spherical harmonics. The coefficients in a multipole expansion are called the multipole moments. Multipole expansions are especially useful very far away from the source because the lowest order terms can be used to approximate the electromagnetic field in that region. Furthermore, the multipole moments provide a physical interpretation of the source: the monopole moment represents the total charge of the system, the dipole moment describes the separation of positive and negative charges and the higher order multipole moments describe the distribution of charge in more complex geometric configurations.

In general relativity, we also want to be able to approximate the gravitational field far away from the source and to interpret the gravitational field. Luckily, there also exist multipole moments in general relativity! However, they are surprisingly difficult to define and we cannot define them in all spacetimes. For stationary asymptotically flat vacuum spacetimes, there exist multiple definitions of multipole moments. Stationary spacetimes are seen as equilibrium states where geodesic motions are permanent. The sources are time-independent. Asymptotic flatness implies the gravitational field falls off far away from the source, allowing us to expand the gravitational field at "infinity". The vacuum condition can in principle be extended to broader classes of spacetimes, including matter. However, there are no multipole moments describing completely arbitrary matter [77]. One can also define multipole moments with arbitrary time dependence, but such definitions often describe only linear perturbations [57, 69, 107, 114]. There are no exact multipole moments to describe the full gravitational field in arbitrary spacetimes. In this thesis, we study multipole moments in stationary asymptotically flat spacetimes describing the full (nonlinear) gravitational field.

In the rest of the introduction, we want to put gravitational multipole moments in perspective. We start with a discussion on multipole moments in Newtonian gravity, after which we turn to general relativity. First, we discuss several attempts to define multipole moments in stationary asymptotically flat vacuum spacetimes and after that we do the same in solutions of the Einstein equations with matter. Multipole moments are not only a theoretical concept, but they can also be measured as we will discuss next. Afterwards, we list the goals of this thesis, provide an outline and list the conventions. We conclude with some guidelines for reading and a list of new results.

### Multipole moments in Newtonian gravity

In this part, we discuss multipole moments in Newtonian gravity. We work in the classical setting of a three-dimensional space. For a more elaborate discussion, one may have a look at the book by Poisson and Will [91, Section 1.5]. In classical mechanics, the gravitational field is described by the Newtonian potential. Outside the mass distribution, the Newtonian potential $V$ is a solution of the Laplace equation:

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \varphi^2} = 0.$$

Separation of variables shows that $V$ can be decomposed as

$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} I^{lm} r^{-(l+1)} Y^{lm}(\theta,\varphi) + \sum_{l=0}^{\infty}\sum_{m=-l}^{l} E^{lm} r^{l} Y^{lm}(\theta,\varphi),$$

for some constants $I^{lm}$ and $E^{lm}$, where the functions $Y^{lm}$ are spherical harmonics [45]. The coefficients $I^{lm}$ are called the internal multipole moments and the coefficients $E^{lm}$ are called the external multipole moments [121]. In many physical applications, we only have either internal or external multipole moments. If there is an external gravitational field, then we want the potential to be well-defined at the origin so that we set $I^{lm} = 0$. On the other hand, if there is a mass distribution contained in a ball around the origin and we study the gravitational field outside the ball, then we want the potential to fall off when going to infinity. Therefore, we set $E^{lm} = 0$. In this thesis, we will restrict ourselves to the second case, so we assume $\lim_{r\to\infty} V(r,\theta,\varphi) = 0$ and the Newtonian potential becomes

$$V(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} I^{lm} r^{-(l+1)} Y^{lm}(\theta,\varphi). \tag{1.1}$$

We unambiguously call the coefficients $I^{lm}$ the Newtonian multipole moments and we forget about the external ones. In this equation, we work with spherical coordinates, but we can also use Cartesian coordinates. A coordinate transformation from $(r,\theta,\varphi)$ to Cartesian coordinates $\left(x^1, x^2, x^3\right)$ yields [107]

$$Y^{lm}(\theta,\varphi) = \sum_{a_1,\ldots,a_l=1}^{3} \mathcal{Y}^{lm}_{a_1\ldots a_l} \frac{x^{a_1}\cdots x^{a_l}}{r^l},$$

where $\mathcal{Y}^{lm}_{a_1\ldots a_l}$ is given by

$$\mathcal{Y}^{lm}_{a_1\ldots a_l} = \sum_{j=0}^{\lfloor \frac{l-m}{2} \rfloor} c^{lmj} \left(\delta^1_{(a_1} + i\delta^2_{(a_1}\right)\cdots\left(\delta^1_{a_m} + i\delta^2_{a_m}\right)\delta^3_{a_{m+1}}\cdots\delta^3_{a_{l-2j}}\delta_{a_{l-2j+1}a_{l-2j+2}}\cdots\delta_{a_{l-1}a_l)}, \tag{1.2}$$

with

$$c^{lmj} = (-1)^m \left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2} \frac{(-1)^j}{2^l j!(l-j)!}\frac{(2l-2j)!}{(l-m-2j)!},$$

for $m \geq 0$. For $m < 0$, we have $\mathcal{Y}^{lm}_{a_1 \ldots a_l} = (-1)^m \overline{\mathcal{Y}^{l,-m}_{a_1 \ldots a_l}}$. So, we can alternatively write (1.1) in Cartesian coordinates as

$$V\left(x^1, x^2, x^3\right) = r^{-1} \sum_{l=0}^{\infty} \sum_{a_1, \ldots, a_l = 1}^{3} \mathcal{I}^l_{a_1 \ldots a_l} \frac{x^{a_1} \cdots x^{a_l}}{r^{2l}}, \tag{1.3}$$

where

$$\mathcal{I}^l_{a_1 \ldots a_l} = \sum_{m=-l}^{l} I^{lm} \mathcal{Y}^{lm}_{a_1 \ldots a_l}. \tag{1.4}$$

From (1.2), it is easy to verify that the tensors $\mathcal{Y}^{lm}$ with components $\mathcal{Y}^{lm}_{a_1 \ldots a_l}$ are symmetric and trace-free. Therefore, we can also see $\mathcal{I}^l$ in (1.3) as a symmetric trace-free tensor with components $\mathcal{I}^l_{a_1 \ldots a_l}$. Equation (1.4) can be inverted by [107]

$$I^{lm} = \frac{4\pi l!}{(2l+1)!} \sum_{a_1, \ldots, a_l = 1}^{3} \mathcal{I}^l_{a_1 \ldots a_l} \overline{\mathcal{Y}^{lm}_{a_1 \ldots a_l}}. \tag{1.5}$$

The relations (1.4) and (1.5) show that the coefficients $I^{lm}$ and the tensors $\mathcal{I}^l$ contain the same information. Therefore, we can see the symmetric trace-free tensors $\mathcal{I}^l$ as a Cartesian version of multipole moments and (1.3) is a multipole expansion with these moments.

We can also take a completely different approach. The idea is to view spatial infinity as a point and do a Taylor expansion around this point. Let $\widetilde{r} = r^{-1}$, and consider $\widetilde{V}$ given by

$$\widetilde{V}(\widetilde{r}, \theta, \varphi) = \widetilde{r}^{-1} V\left(\widetilde{r}^{-1}, \theta, \varphi\right).$$

Again, we use Cartesian coordinates but now we perform a coordinate transformation from spherical coordinates $(\widetilde{r}, \theta, \varphi)$ to Cartesian coordinates $\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$. The function $\widetilde{V}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$ with respect to $(\widetilde{r}, \theta, \varphi)$ or, equivalently, $\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$, meaning $\widetilde{\Delta}\widetilde{V} = 0$ where $\widetilde{\Delta}$ is the Laplacian with respect to $(\widetilde{r}, \theta, \varphi)$ (or, equivalently, $\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$) [45]. From our knowledge of $V$, we observe that $\widetilde{V}$ must be finite at $\widetilde{r} = 0$. But then the fact that $\widetilde{V}$ is harmonic on $\mathbb{R}^3 \setminus \{0\}$ implies that it must be harmonic on all of $\mathbb{R}^3$, including the origin [19]. Hence, the potential $\widetilde{V}$ is analytic around $\widetilde{r} = 0$ and a Taylor expansion gives [6]

$$\widetilde{V}\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right) = \sum_{l=0}^{\infty} \sum_{a_1, \ldots, a_l = 1}^{3} \frac{\widetilde{x}^{a_1} \cdots \widetilde{x}^{a_l}}{l!} \left. \frac{\partial^l \widetilde{V}}{\partial \widetilde{x}^{a_1} \cdots \partial \widetilde{x}^{a_l}} \right|_{\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right) = 0}.$$

For each $l$, define a tensor $P^l$ by

$$P^l_{a_1 \ldots a_l} = \frac{\partial^l \widetilde{V}}{\partial \widetilde{x}^{a_1} \cdots \partial \widetilde{x}^{a_l}}.$$

Each $P^l$ is symmetric because the partial derivatives commute and it is trace-free because $\widetilde{\Delta}\widetilde{V} = 0$ and we can commute the contracted partial derivatives to the end. Moreover, we can define the tensors $P^l$ recursively by $P^0 = \widetilde{V}$ and

$$P^{l+1}_{a_1 \ldots a_{l+1}} = \frac{\partial P^l_{a_1 \ldots a_l}}{\partial \widetilde{x}^{a_{l+1}}}. \tag{1.6}$$

We view

$$\widetilde{V}\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right) = \sum_{l=0}^{\infty} \sum_{a_1,\dots,a_l=1}^{3} \frac{\widetilde{x}^{a_1} \cdots \widetilde{x}^{a_l}}{l!} P^l_{a_1\dots a_l}\Big|_{(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)=0}, \tag{1.7}$$

as a multipole expansion of the gravitational field. This gives another set of multipole moments, namely the tensors $P^l\big|_{(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)=0}$. Equations (1.1)/(1.3) and (1.7) provide two multipole expansions for the same gravitational field. The two Cartesian coordinate systems are related by $\widetilde{x}^i = \frac{x^i}{r^2}$, so (1.3) reads

$$\widetilde{V}\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right) = \sum_{l=0}^{\infty} \sum_{a_1,\dots,a_l=1}^{3} \mathcal{I}^l_{a_1\dots a_l} \widetilde{x}^{a_1} \cdots \widetilde{x}^{a_l}.$$

Comparing to (1.7) shows that

$$P_{a_1\dots a_l}\big|_{(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)=0} = l! \mathcal{I}^l_{a_1\dots a_l}. \tag{1.8}$$

Therefore, we see that the two multipole expansions are equivalent, even though the derivations are very different.

### Multipole moments in stationary vacuum solutions

For compact objects, Newtonian gravity is usually not sufficient and we need to turn to general relativity. Therefore, we want to define relativistic multipole moments, but that is much more difficult. The origins of research on multipole moments in general relativity date back to the sixties and seventies. In stationary asymptotically flat vacuum spacetimes, several attempts to define relativistic multipole moments have been made since then. We can distinguish the existing definitions by whether they are coordinate-dependent of coordinate-independent. In this part, we mention some of the definitions.

First, we mention some coordinate-dependent definitions. In 1968, Van der Burg [109] defined multipole moments based on picking a "suitable radial coordinate" and performing a power expansion. However, the "suitable radial coordinate" is not invariant and there is no method to calculate it for an arbitrary spacetime [93]. A definition for multipole moments in static spacetimes was given by Clarke and Sciama [31] in 1971. Here, the idea is to use Sommerfeld's method [23] to solve a Poisson equation. It is not known whether these multipole moments can be generalised to stationary spacetimes [93]. Another coordinate-dependent definition of multipole moments in stationary spacetimes was given by Thorne [107] in 1980. The criterion for the coordinates was proposed and discussed by Thorne and it is clear that they give well-defined multipole moments. Thorne's multipole moments are also used in applications because they are relatively easy to calculate (for low orders).

A coordinate-independent definition for multipole moments was provided by Geroch [42] in 1970, but it only works for static spacetimes. The approach by Geroch was generalised to stationary spacetimes by Hansen [48] in 1974, delivering two sets of multipole moments: the mass and angular momentum multipole moments. They should be seen as the mass (mass monopole moment) and angular momentum (angular momentum dipole moment) of the system and the higher order moments represent higher order corrections. The main advantage of the definitions given by Geroch and Hansen is that they are purely geometric and independent of the chosen coordinates. Alternatively, one can also view the Geroch–Hansen multipole moments as power expansions of some gravitational potentials, as shown by Beig and Simon [10, 11, 103] in 1981. In fact, Beig and Simon introduced the power expansions as an alternative definition for multipole moments and showed they are equivalent to the ones by Geroch and Hansen.

Due to their invariant nature and the fact that they generalise to stationary spacetimes, the definitions by Geroch–Hansen and Thorne are the most important ones [93]. One can also add the definition by Beig–Simon to the list, but these multipole moments are just another way of writing the multipole moments by Geroch–Hansen. Surprisingly, the multipole moments defined by Thorne are also equivalent to the ones by Geroch–Hansen, as shown by Gürsel [46] in 1983. One can see the Thorne formalism as the relativistic version of the multipole moments in (1.1), while the Geroch–Hansen formalism gives a relativistic version of the multipole moments in (1.7) using the recursion relation (1.6). The equivalence of the Thorne and Geroch–Hansen formalism can be interpreted in the same way as how these two definition for Newtonian multipole moments are equivalent. Both formalisms and their equivalence will be discussed in Part II of this thesis. Even though research on multipole moments started about 60 years ago, it is surprisingly hard to get our definitions and assumptions straight. In this thesis, we want to pay special attention to such issues and fill in some of the gaps in the constructions of multipole moments.

### Multipole moments in stationary spacetimes with matter

The multipole moments above apply to vacuum solutions of the Einstein equations, but we also want multipole moments in solutions with matter. The Geroch–Hansen formalism was generalised to Einstein–Maxwell solutions by Simon [102] in 1984. In this setting, there are not only the gravitational mass and angular momentum multipole moments, but we also have electric and magnetic multipole moments. The electromagnetic multipole moments are needed to distinguish solutions of the Einstein–Maxwell equations based on their multipole moments. For example, the Kerr and Kerr–Newman solutions have the same gravitational multipole moments, but the electromagnetic multipole moments are different.

The Thorne formalism has not been generalised to Einstein–Maxwell solutions so far. To calculate the first few multipole moments, the Thorne formalism is often easier than the Geroch–Hansen formalism. Therefore, it would be convenient to generalise the Thorne formalism to Einstein–Maxwell solutions. We achieve such a generalization in Chapter 8.

If we allow for arbitrary matter fields, there is still a lot unkown. A geometric way to define the gravitational multipole moments in stationary asymptotically flat spacetimes in presence of matter is given by Mayerson [77] in 2023. The Geroch–Hansen formalism relies on the closedness of the so-called twist one-form, but it does not need to be closed anymore in non-vacuum solutions. Mayerson proposed an alternative twist one-form that is closed and reduces to the ordinary one in vacuum. This allows us to define multipole moments in the same way as in the Geroch–Hansen formalism. However, the resulting multipole moments only describe the gravitational field. Ultimately, we want to complement the gravitational multipole moments by multipole moments containing information about the matter. We investigate this issue for scalar field solutions of the Einstein equations in Chapter 9.

It can be seen as an ultimate goal to define multipole moments in all stationary asymptotically flat spacetimes. From a physical point of view, the Kerr and Kerr–Newman spacetimes are probably the most important ones and we are luckily able to calculate their multipole moments. Scalar field solutions for the hypothetical boson stars [99]. It could also be interesting to investigate Einstein–Yang–Mills theory, where we have non-abelian matter fields. In this theory, there are black holes that do not only depend on mass, charge and angular momentum [21, 68]. Einstein–Yang–Mills theory is the non-abelian generalisation of the Einstein–Maxwell theory.

### Measuring multipole moments

In the discussion above and in this thesis, multipole moments are mainly a theoretical concept describing the gravitational field. However, they also have concrete physical applications. Multipole moments provide a way to test general relativity and identify sources. Instead of determining the multipole moments for a theoretical spacetime, we now want to determine the multipole moments for a real-world source.

In the framework of Newtonian gravity, the Earh's multipole moments have been measured by various satellite projects such as GOCE, LAGEOS and GRACE [30, 34, 111]. The measurements are done by precise tracking of the satellites orbiting around the earth. In the GRACE project, they managed to calculate the Earth's multipole moments up to order $l \sim 360$ [91].

However, for compact objects, general relativity is the appropriate framework. Measuring multipole moments in general relativity has been pioneered by Ryan [96, 97] in the nineties. The multipole moments of a large compact object at the center can be determined by the gravitational radiation emitted by a smaller compact object orbiting around it. In the case of extreme mass ratio inspirals, the planned gravitational wave detector LISA is potentially able to measure the first few multipole moments [5, 8, 63, 74, 94]. It should be possible to get rather accurate measurements for the first three multipole moments. For higher orders, we would lose precision and it depends on the situation [39].[1]

Like was done with a satellite around the Earth, multipole moments of large compact objects can also be determined by measuring the motion of stars or pulsars around them [113]. Alternative methods are studying accretion disks [22] and analysing the shadows of black holes by the event horizon telescope [92].

Measuring multipole moments allows for several tests in general relativity. For example, we can experimentally test the no-hair theorem/conjecture, which states that stationary black hole solutions in vacuum are completely characterised by its mass and angular momentum. (If we allow for the presence of an electromagnetic field and the spacetime should solve the Einstein–Maxwell equations, we also need charge.) The no-hair conjecture has not been proven yet, although it is often referred to as a theorem. Some weak versions are proven, but they require rather strong assumptions, see for example [70, Theorem 10.26]. The measurements allow us to check whether the multipole moments are consistent with the ones for the Kerr (or Kerr-Newman) solution of the Einstein equations, providing a real-world test for the no-hair theorem/conjecture [24, 60, 122]. The higher order multipole moments for the Kerr

[1] The order to which the multipole moments can be calculated depends on the model, the masses and radii of the compact objects and the signal-to-noise ratio. Several tables describing the accuracy of the measurements using a model by Ryan can be found in [97].

(or Kerr-Newman) solution can be expressed in terms of the monopole and dipole moments. Therefore, already measuring up to the quadrupole moment would provide a consistency check.

Another application where multipole moments contain useful information is for neutron stars. The innermost stable circular orbit (ISCO) marks the inner edge of the accretion disks and one can relate properties of the ISCO to the multipole moments [16, 98, 101]. There is also a no-hair relations for neutron stars. The higher order moments should depend only on the monopole, dipole and quadrupole moments [118], so measuring the octopole moment gives a consistency check.

### Goals of this thesis

In this thesis, we focus on the theoretical side of the medal. The goal is three-fold:

- Reviewing the important constructions for multipole moments in stationary asymptotically flat spacetimes;
- Filling in the (mathematical) gaps in their constructions;
- Extending the definitions to broader classes of solutions of the Einstein equations with matter.

### Outline

The thesis is divided into three parts. In Part I, we discuss the geometric setting in which we define multipole moments. In particular, we discuss stationarity in Chapter 2 and asymptotic flatness in Chapter 3. This part is the cornerstone of the thesis in the sense that we need to make precise what assumptions have been made to define multipole moments.

The important constructions for multipole moments in vacuum solutions of the Einstein equations are described in Part II. In particular, we discuss the Geroch–Hansen formalism in Chapter 4 and the Thorne formalism in Chapter 5. The equivalence of the resulting multipole moments is shown in Chapter 6, together with some important properties.

In Part III, multipole moments in solutions of the Einstein equations with matter are discussed. We start with a discussion on the construction for multipole moments in Einstein–Maxwell solutions by Simon in Chapter 7. Simon's approach is a natural generalisation of the Geroch–Hansen formalism. In Chapter 8, we propose an alternative approach to define multipole moments in the presence of an electromagnetic field by mimicking the approach by Thorne. We show that this also gives an equivalent set of multipole moments to those defined by Simon as we would expect from vacuum. To open the door to other matter fields, we discuss scalar field solutions in Chapter 9. This thesis is finalised with conclusions and an outlook in Chapter 10.

### Conventions and notation

We use geometrised units, effectively meaning that the speed of light $c$ and the gravitational constant $G$ are both set to 1. Moreover, we assume spacetime is a smooth, orientable, connected four-dimensional manifold with empty boundary and endowed with a Lorentzian metric. We adopt the "mostly plus" convention, so the signature of the metric is $(-+++)$.

We use the Einstein summation convention. Greek indices belong to and sum over $0, 1, 2, 3$, while Latin indices only belong to and sum over $1, 2, 3$. Unless explicitly stated otherwise, all our manifolds and tensor fields are smooth. We assume a spacetime is stationary unless explicitly stated otherwise and we assume it is asymptotically flat from Chapter 4 onwards, unless explicitly stated otherwise. The Levi-Civita connection for a 4-dimensional spacetime $(M, g)$ is denoted by $\nabla$ and the Levi-Civita connection for a 3-dimensional space $(S, h)$ is denoted by $D$. If we work on a 3-dimensional space $\left(\widetilde{S}, \widetilde{h}\right)$, we also decorate the Levi-Civita connection so that it is denoted by $\widetilde{D}$.

In this thesis, we mostly use more global notation in differential geometry used by mathematicians, following Lee [72, 73] and O'Neill [86]. Table 1.1 is a non-exhaustive list relating the notations by mathematicians and physicists such as Wald [112]. It is an extended version of the table in Natário's book [83, Section 1.2].

Table 1.1: Notation by mathematicians versus physicists in general relativity.

| **Object** | **Mathematicians** | **Physicists** |
|---|---|---|
| Vector field | $X$ | $X^\mu$ |
| Covector field/One-form | $\omega$ | $\omega_\mu$ |
| $(k, l)$-Tensor field | $T$ | $T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$ |
| Metric tensor | $g(\cdot, \cdot)$ | $g_{\mu\nu}$ |
| Tensor product | $S \otimes T$ | $S^{\mu_1 \dots \mu_{k_1}}_{\nu_1 \dots \nu_{l_1}} T^{\mu_{k_1+1} \dots \mu_{k_1+k_2}}_{\nu_{l_1+1} \dots \nu_{l_1+l_2}}$ |
| Lowered vector field | $X^\flat$ | $X_\mu$ |
| Raised covector field | $\omega^\sharp$ | $\omega^\mu$ |
| Lowering an index | $\downarrow^i_j T$ | $T^{\mu_1 \dots \mu_{k-1}}_{\nu_1 \dots \nu_{l+1}} = g_{\nu_j \rho} T^{\mu_1 \dots \mu_{i-1} \rho \mu_i \dots \mu_{k-1}}_{\nu_1 \dots \nu_{j-1} \nu_{j+1} \dots \nu_{l+1}}$ |
| Raising an index | $\uparrow^i_j T$ | $T^{\mu_1 \dots \mu_{k+1}}_{\nu_1 \dots \nu_{l-1}} = g^{\mu_i \rho} T^{\mu_1 \dots \mu_{i-1} \mu_{i+1} \dots \mu_{k+1}}_{\nu_1 \dots \nu_{j-1} \rho \nu_j \dots \nu_{l-1}}$ |
| Contraction | $C^i_j(T)$ | $T^{\mu_1 \dots \mu_{i-1} \rho \mu_i \dots \mu_{k-1}}_{\nu_1 \dots \nu_{j-1} \rho \nu_j \dots \nu_{l-1}}$ |
| Contractions with the metric | $C_{ij}(T)$ | $g^{\rho\sigma} T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_{i-1} \rho \nu_i \dots \nu_{j-2} \sigma \nu_{j-1} \dots \nu_l}$ |
| Covariant derivative | $\nabla_Y X$ | $Y^\mu \nabla_\mu X^\nu$ |
| Total covariant derivative | $\nabla X$ | $\nabla_\mu X^\nu$ |
| Lie derivative | $\mathcal{L}_X T$ | $\mathcal{L}_X T^{\mu_1 \dots \mu_k}_{\nu_1 \dots \nu_l}$ |

## Guidelines for reading

Since this thesis is both a master's thesis in Mathematics and in Physics and Astronomy, we give some guidance for reading this thesis. Both mathematics and physics are intimately woven together in this thesis, so it is difficult to distinguish them. We also require quite some knowledge from both areas. In particular, one should be familiar with differential geometry, general relativity, spherical harmonics and partial differential equations.

From a mathematical perspective, the geometric Geroch–Hansen formalism is most appealing. For a rigorous approach to geometric multipole moments in full generality including matter, one would have to read Chapters 2, 3 (possibly except Section 3.3), 4, 7 and 9. The background of these parts lie mainly in Lorentzian geometry and the theory of (elliptic) partial differential equations. In the Thorne formalism, spherical harmonics play a central role. They appear in Chapters 5, 6 and 8 and very briefly in Chapter 9.

On the other hand, one may not need all (mathematical) details to understand the main ideas behind the constructions of multipole moments. For that purpose, it is most likely sufficient to read Chapter 2 up to Theorem 2.3 and Proposition 2.6, Section 3.1 and from Chapter 4 onwards. In Section 4.3 and Section 7.2, a method to calculate multipole moments in axisymmetric spacetimes is discussed. They can be skipped if one is not interested in calculating multipole moments in such spacetimes.

### List of new results

Since this thesis is reviewing, filling in gaps and developing some new parts, it can be difficult distinguish the nature of each result. For most proofs in this thesis, they are either given for completeness or to reformulate/clarify/simplify some parts. However, there are also some new results and we correct a few mistakes in the original papers. The results of the latter nature can be found in the following list:

- The proof of Proposition 2.8 is slightly different from the original paper by Garfinkle and Harris [40] that may solve a mistake. See footnote 2 for more information.
- Theorem 3.3 is a correction of the wrong result by Geroch [42], see footnote 5.
- Proposition 4.10 and Corollary 4.11 generalise results by Beig [11].
- The multipole moments defined in Definition 7.6 can be seen as a slight generalisation for electrovacuum where multipole moments are originally defined by Simon [102] because we do not assume that the electromagnetic field is exact.
- In Chapter 8, we develop a new way to define multipole moments using physical intuition. It is not supposed to be rigorous mathematical result.
- Everything, most notably Definition 9.3, in Section 9.2 is new.

# Part I

# Geometric Setting

# Chapter 2

# Stationary Spacetimes

We want our multipole moments to give a set of constants, independent of time. Therefore, we restrict ourselves to equilibrium states. Such spacetimes are called stationary, and consist of permanent geodesic motions. The goal of this chapter is to recall the definition and investigate important properties of stationary spacetimes. The definition is recalled in Section 2.1. Time symmetry allows us to quotient by the time direction and we want to analyse the resulting three-dimensional space. This space is the so-called observer space, and is constructed in Section 2.2. In Section 2.3, we define the twist covector field and discuss some important properties. The twist covector field can be used to describe the dynamics of stationary spacetimes. It allows us to reduce the Einstein equations to the three-dimensional observer space, which is discussed in Section 2.4.

## 2.1 Definition of stationary spacetimes

In this section, we want to recall the definition of stationary spacetimes. The common definition for stationary spacetimes is that the spacetime must admit a timelike Killing vector field. However, it depends on the source whether to assume this vector field is complete or not. For example, completeness of the timelike Killing vector field is assumed in [70, 112], while it is not in [25, 49, 90]. The difference between the two definitions is whether vector field generates a one-parameter group action (or a global flow) on the manifold or not. Since this one-parameter group action helps us to reduce our spacetime to a lower-dimensional space, we include the completeness assumption.

**Definition 2.1.** A Lorentzian manifold is called *stationary* if it has a complete timelike Killing vector field. Such a complete timelike Killing vector field is also called a *stationary vector field*.

Such a stationary vector field is, in particular, a timelike vector field and induces a time orientation. So, if we fix a complete timelike Killing vector field, we can always assume it defines the time orientation and is future-directed.

Locally, the fact that a timelike vector field is nonvanishing implies that there is a coordinate system with a coordinate $t$ such that $\frac{\partial}{\partial t}$ coincides with this vector field. If it is a Killing vector field, the component functions of the metric tensor are independent of $t$. Conversely, if we have a coordinate system such that the component functions of the metric tensor are independent of a coordinate $t$, then $\frac{\partial}{\partial t}$ is a Killing vector on the domain of the coordinate chart. To get a global vector field it should be checked that the local Killing vector fields glue together on the overlap of two charts. The completeness assumption cannot be checked locally.

The one-parameter group action induced by a (complete) stationary vector field has important consequences. More on this in the next section, but one of them is that a stationary spacetime is reflecting [78, Theorem 4.10]. That is, we have $I^-(p) \subseteq I^-(q)$ if and only if $I^+(q) \subseteq I^+(p)$ for every two points $p$ and $q$. Here, $I^+(p)$ is the subset of $M$ consisting of points that can be reached by future-directed timelike curves starting at $p$ and $I^-(p)$ is the subset of $M$ consisting of points that can be reached by past-directed timelike curves starting at $p$.

## 2.2 Observer space

In this section, we discuss the observer space, which is found by taking out the time direction. First, we discuss the construction. We see that the observer space turns the stationary spacetime into a principal $\mathbb{R}$-bundle, which carries a natural connection. For globally hyperbolic stationary spacetimes, a smooth, spacelike Cauchy surface turns out to be diffeomorphic to the observer space. At the end of this section, we identify the tensor fields on the observer space with tensor fields on the spacetime and discuss three possible Riemannian metrics on the observer space. Recall that we assume any spacetime is stationary and we let $\xi$ be a stationary vector field.

### Construction of the observer space

The one-parameter group action induced by a stationary vector field gives an $\mathbb{R}$-action on $M$ and we can take the quotient. To ensure this space will be a manifold, we need another assumption. A spacetime is chronological if there are no closed timelike curves. In the setting of a reflecting spacetime, we can weaken this condition by demanding there exists a point through which there is no closed timelike curve. Such a spacetime is called non-totally vicious. We typically assume a spacetime is connected, and then a spacetime that is not non-totally vicious would admit a closed timelike curve through any pair of two points [78]. In Theorem 2.3, we use the chronology condition, but we can equivalently assume that the spacetime is non-totally vicious. The non-totally vicious condition is the lowest level on the causal ladder [78], so it is quite remarkable we only need such a weak causality condition.

The one-parameter group action allows us to consider the orbits of this action, which are the maximal integral curves of this fixed stationary vector field, seen as sets. The three-dimensional space we want to consider is the set of maximal integral curves.

**Definition 2.2.** Let $(M, g)$ denote a stationary spacetime with a stationary vector field $\xi$ whose global flow is $\theta$. The *observer space* of $(M, g)$ is the quotient of $M$ under the $\mathbb{R}$-group action $\theta$. We denote the observer space of $(M, g)$ by $S$.

It is called the observer space because an observer can move along the maximal integral curves of $\xi$ and reach all points in $M$. To each point $p \in M$, we can assign the unique maximal integral curve of $\xi$ through $p$, defining a map $\pi\colon M \to S$. This map is clearly surjective as every integral curve goes through at least one point. It is the quotient map when viewing the observer space as a quotient under the one-parameter group action. In the category of sets or topological spaces, it is clear that this works but quotients are not automatically smooth manifolds. The goal of Theorem 2.3 is to prove that the set $S$ is a smooth manifold and $\pi$ is a surjective smooth submersion, giving a quotient in the category of smooth manifolds. In [42], it is basically assumed that this works. Locally, the situation is not too bad and the chronology condition may even be dropped, but for a global result it will be very useful.

**Theorem 2.3.** *Let $(M, g)$ be a chronological, stationary spacetime with a complete timelike Killing vector field $\xi$. Let $S$ be the observer space, then $S$ can be given a three-dimensional smooth manifold structure such that the map $\pi\colon M \to S$, mapping a point $p \in M$ to the maximal integral curve of $\xi$ through $p$, is a smooth surjective submersion.*

*Proof.* Since $\xi$ is complete, it has a global flow $\theta\colon \mathbb{R} \times M \to M$ which defines a smooth $\mathbb{R}$-action on $M$ by $t \cdot p = \theta(t, p)$. Each curve $\theta^{(p)} = \theta(\cdot, p)$ is a maximal integral curve of $\xi$, in particular it is a future-directed timelike curve because $\xi$ is timelike. We want the action to be free, meaning we need that for each $p \in M$, $t \cdot p = p$ implies $t = 0$. Suppose $t_1 \cdot p = t_2 \cdot p$, then without loss of generality we can assume that $t_1 \leq t_2$, otherwise swap $t_1$ and $t_2$. Suppose $t_1 < t_2$, then $\theta^{(p)}\big|_{[t_1,t_2]}\colon [t_1, t_2] \to M$ is a closed timelike curve, which is a contradiction with the chronology condition. Therefore, we must have $t_1 = t_2$ and the action is free.

So, we have a group acting smoothly and freely, and we also want it to act properly. To prove it, we follow [33, pp. 1646–1647]. We use the characterization of a proper action using sequence, which reads that the action is proper if for all sequences $(p_i)_{i\in\mathbb{N}}$ in $M$ and $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that both $(p_i)_{i\in\mathbb{N}}$ and $(t_i \cdot p_i)_{i\in\mathbb{N}}$ converge, $(t_i)_{i\in\mathbb{N}}$ has a convergent subsequence. Suppose the action is not proper, then we can take a sequence $(p_i)_{i\in\mathbb{N}}$ in $M$ and a sequence $(t_i)_{i\in\mathbb{N}}$ in $\mathbb{R}$ such that $(p_i)_{i\in\mathbb{N}}$ and $(t_i \cdot p_i)_{i\in\mathbb{N}}$ converge, but $(t_i)_{i\in\mathbb{N}}$ does not have a convergent subsequence. Take $p, q \in M$ such that $p_i \to p$ and $t_i \cdot p_i \to q$ as $i \to \infty$. By the Bolzano–Weierstrass theorem, the sequence $(t_i)_{i\in\mathbb{N}}$ is unbounded. Without loss of generality, we can assume that $t_i \to \infty$ by passing to a subsequence and the case that it diverges to $-\infty$ can be discussed in the same way.

Let $t \in \mathbb{R}$ and $\varepsilon > 0$ be arbitrary. We write $p \ll q$ if there is a future-directed timelike curve from $p$ to $q$. Then $t \cdot p \ll (t + \varepsilon) \cdot p$ because $\theta^{(p)}$ is a timelike curve. Since the chronological relation $I$ is open, we can take an open neighborhood $U$ of $(t + \varepsilon) \cdot p$ in $M$ such that $U \subseteq I^+(t \cdot p)$. Also, $(-\varepsilon) \cdot q \ll q$, so we can take an open neighborhood $V$ of $(-\varepsilon) \cdot q$ in $M$ such that $V \subseteq I^-(q)$. By taking $i_0 \in \mathbb{N}$ large enough, we have $t_i > t + 2\varepsilon$, $(t + \varepsilon) \cdot p_i \in U$ and $(t_i - \varepsilon) \cdot p_i \in V$ for all $i \geq i_0$. This gives

$$t \cdot p \ll (t + \varepsilon) \cdot p_i \ll (t_i - \varepsilon) \cdot p_i \ll q,$$

where we used $(t + \varepsilon) \cdot p_i \in U$ in the first relation, $t_i > t + 2\varepsilon$ in the second relation, and $(t_i - \varepsilon) \cdot p_i \in V$ in the last relation. So $t \cdot p \in I^-(q)$ and since $t \in \mathbb{R}$ is arbitrary, we find that the whole maximal integral curve of $\xi$ through $p$ is contained in $I^-(q)$.

To arrive at a contradiction, we follow [51, p. 34]. Since a spacetime is connected, there exists a smooth curve $\sigma\colon [0, 1] \to M$ such that $\sigma(0) = q$ and $\sigma(1) = p$. Let $\alpha\colon \mathbb{R} \times [0, 1] \to M$ be defined by $\alpha(t, s) = t \cdot \sigma(s)$, then $d\alpha\big(\frac{\partial}{\partial t}\big) = \xi \circ \alpha$ and we define $X = d\alpha\big(\frac{\partial}{\partial s}\big)$. We can take a constant $c > 0$ large enough such that the vector $c\xi_{\alpha(0,s)} + X_{(0,s)}$ is timelike and future-directed for all $s \in [0, 1]$. Note that we use compactness of $[0, 1]$ for this constant $c$ to exist.

Then,
$$c\xi_{\alpha(t,s)} + X_{(t,s)} = d(\theta_t)\big(c\xi_{\alpha(0,s)} + X_{(0,s)}\big),$$
where we used that $(\theta_t \circ \alpha)(t', s) = \alpha(t' + t, s)$, so $d(\theta_t)d\alpha_{(0,s)} = d(\theta_t \circ \alpha)_{(0,s)} = d\alpha_{(t,s)}$. Since $\xi$ is a Killing vector field, $\theta_t$ is an isometry, preserving the causal structure. Therefore, $c\xi_{\alpha(t,s)} + X_{(t,s)}$ is still future-directed and timelike for all $t \in \mathbb{R}$. Let $\gamma$ be the integral curve of $c\xi \circ \alpha + X$ starting at $q$, then $\gamma(t) = \alpha(ct, t)$ for $t \geq 0$. So, $\gamma$ is a future-directed timelike curve from $\gamma(0) = \sigma(0) = q$ to $\gamma(1) = c \cdot p$. Hence, $c \cdot p \in I^+(q)$, but above we found that $c \cdot p \in I^-(q)$. But then there is a closed timelike curve by concatenating the future-directed timelike curves from $c \cdot p$ to $q$ and from $q$ to $c \cdot p$. This contradicts the chronology condition, so the action must be proper.

Therefore, $M/\mathbb{R}$ can be given a unique smooth manifold structure such that the corresponding quotient map is a smooth surjective submersion [72, Theorem 21.10]. Here, $M/\mathbb{R}$ consist precisely of the orbits of the flow, which are the maximal integral curves of $\xi$. So, $S = M/\mathbb{R}$ is a smooth manifold and the quotient map $\pi\colon M \to S$ is a smooth surjective submersion. Finally, the dimension of $S$ is $\dim S = \dim M - \dim \mathbb{R} = 4 - 1 = 3$. □

A more general version of this theorem is proven by Harris [49, Theorem 1]. He works with a conformal Killing vector field instead of a Killing vector field. One can even drop the "Killing" assumption, then we end up with a so-called near-manifold. A near-manifold is a topological space that is locally Euclidean and second-countable, but it does not have to be Hausdorff [49, Theorem 2].

In the proof of the theorem above, our choice for a complete timelike Killing vector field becomes clear. We make crucial use of the one-parameter group action induced by $\xi$. We want to understand this completeness assumption a bit better. By the uniform time lemma, a vector field is complete if and only if there exists $\varepsilon > 0$ such that the domain of the flow line starting at $p$ contains $(-\varepsilon, \varepsilon)$ for all $p \in M$ [72, Lemma 9.15]. The crucial part in this result is that $\varepsilon$ does not depend on the point $p$. In a vacuum, maximal globally hyperbolic spacetime, it suffices to check this around a Cauchy surface. More precisely, in a vacuum, maximal globally hyperbolic spacetime with Cauchy surface $\Sigma$, a Killing vector field $X$ is complete if and only if there exists $\varepsilon > 0$ such that the flow $\theta(t, p)$ of $X$ is defined for all $t \in (-\varepsilon, \varepsilon)$ and $p \in \Sigma$ [29, Theorem 1.1]. This result can be applied to the vector field $\xi$. Other assumptions that imply completeness of a timelike Killing vector field in a stationary spacetime are timelike and null geodesic completeness [40, Lemma 1].

### Stationary spacetimes as principal $\mathbb{R}$-bundles

The proof of Theorem 2.3 also implies that $\pi\colon M \to S$ is a principal $\mathbb{R}$-bundle. We want to understand this principal bundle a bit better. Since the fiber of the principal bundle is a Euclidean space, it is trivial [65, Theorem I.5.7]. The proof of this result relies on algebraic topology. It means that there is an isomorphism of principal $\mathbb{R}$-bundles between $\pi\colon M \to S$ and the projection $\mathbb{R} \times S \to S$. Alternatively, we can view it as a global section of $\pi$. The Lorentzian metric $g$ on $M$ naturally endows the principal bundle with a connection.

**Proposition 2.4.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$ and observer space $S$. Let $\pi\colon M \to S$ denote the canonical projection. Then the orthogonal complement of $\operatorname{Ker} d\pi$ with respect to $g$ is a connection on the principal $\mathbb{R}$-bundle $\pi\colon M \to S$.*

*Proof.* We already have the metric $g$ on $M$. Let $\theta_t$ denote the action by $t \in \mathbb{R}$, then $(\theta_t)^* g = g$ because $\xi$ is a Killing vector field, so the flow acts by isometries. Since $d\pi_p$ vanishes on $\xi_p$, it clearly restricts to an isomorphism between $(\operatorname{Ker} d\pi_p)^\perp$ and $T_pS$, so $(\operatorname{Ker} d\pi)^\perp$ is a horizontal distribution. Let $v \in (\operatorname{Ker} d\pi_p)^\perp$, then $g_p(v, \xi_p) = 0$. Since $\theta$ is the flow of $\xi$,

$$d(\theta_t)_p(\xi_p) = d(\theta_t)_p\left(\left(\theta^{(p)}\right)'(0)\right) = \frac{d}{ds}\bigg|_{s=0} \theta(t, \theta(s,p)) = \frac{d}{ds}\bigg|_{s=0} \theta(s, \theta(t,p)) = \xi_{t\cdot p}.$$

Therefore, we have

$$g_{t\cdot p}(d(\theta_t)_p(v), \xi_{t\cdot p}) = ((\theta_t)^* g)_p(v, \xi_p) = g_p(v, \xi_p) = 0,$$

so $d(\theta_t)_p(v) \in (\operatorname{Ker} d\pi_p)^\perp$. Hence, $d(\theta_t)_p$ restricts to a linear map between $(\operatorname{Ker} d\pi_p)^\perp$ and $(\operatorname{Ker} d\pi_{t\cdot p})^\perp$, which is injective because $d(\theta_t)_p$ is an isomorphism between $T_pM$ and $T_{t\cdot p}M$. By dimensionality, $d(\theta_t)_p \colon (\operatorname{Ker} d\pi_p)^\perp \to (\operatorname{Ker} d\pi_{t\cdot p})^\perp$ is an isomorphism, so $(\operatorname{Ker} d\pi)^\perp$ is an invariant distribution. Hence, $(\operatorname{Ker} d\pi)^\perp$ defines a connection on $\pi \colon M \to S$. □

Alternatively, a connection can also be defined by a one-form. In that case, we want to consider the one-form $\alpha = -\lambda^{-1}\xi^\flat$ on $M$, where $\lambda = -g(\xi, \xi)$. Clearly, $\operatorname{Ker} \xi_p^\flat = (\operatorname{Ker} d\pi_p)^\perp$, showing this one-form corresponds to the horizontal distribution $(\operatorname{Ker} d\pi_p)^\perp$. To see that $\alpha$ is indeed a connection on the principal $\mathbb{R}$-bundle $\pi \colon M \to S$, we need that it is invariant under the group action and reproduces the Lie algebra generators of the infinitesimal action. Since $\xi$ is a Killing vector field, we have $\mathcal{L}_\xi g = 0$ and we trivially have $\mathcal{L}_\xi \xi = [\xi, \xi] = 0$, so we see that $\mathcal{L}_\xi \lambda = 0$. Moreover, raising and lowering indices commutes with $\mathcal{L}_\xi$ because $\xi$ is a Killing vector field, so we also have $\mathcal{L}_\xi(\xi^\flat) = (\mathcal{L}_\xi \xi)^\flat = 0$. But then we see that $\mathcal{L}_\xi \alpha = 0$, so $(\theta_t)^*\alpha = \alpha$ and $\alpha$ is invariant under the group action. The infinitesimal action of $\theta$ is $\rho \colon \mathbb{R} \to \mathfrak{X}(M)$ given by

$$\rho(X)_p = d\left(\theta^{(p)}\right)_0 \left(X \frac{d}{dt}\bigg|_{t=0}\right) = X\left(\theta^{(p)}\right)'(0) = X\xi_p,$$

so

$$i_{\rho(X)}\alpha = -\lambda^{-1} g(\xi, X\xi) = X\frac{g(\xi,\xi)}{g(\xi,\xi)} = X.$$

Therefore, $\alpha$ is indeed a connection one-form for our principal $\mathbb{R}$-bundle, so it is the connection one-form corresponding to the connection from Proposition 2.4. Since we assume we are given a metric $g$ on $M$, the connection is canonical in the sense that it arises naturally from the metric.

The canonical connection also comes with its related curvature. Since $\mathbb{R}$ is an abelian group, the curvature is just $d\alpha \in \Omega^2(M)$, where $\alpha = -\lambda^{-1}\xi^\flat$. We know from the curvature of a principal bundle that it is a basic two-form, so it reduces to a two-form $K$ on $S$ satisfying $\pi^* K = d\alpha$. We can also check directly that $d\alpha$ lives on $S$. Since the Lie derivative commutes with the exterior derivative, we have

$$\mathcal{L}_\xi d\alpha = d\mathcal{L}_\xi \alpha = 0.$$

We also want to calculate contractions of $d\alpha$ with $\xi$. By antisymmetry of $d\alpha$ it suffices to consider only the contraction in the first index, for which we have

$$(i_\xi d\alpha)(X) = d\alpha(\xi, X) = \xi(\alpha(X)) - X(\alpha(\xi)) - \alpha([\xi, X]).$$

For the second term, we note that $\alpha(\xi) = 1$ is constant, so $X(\alpha(\xi)) = 0$. Using that $\mathcal{L}_\xi \lambda = 0$ and torsion-freeness and metric-compatibility of the Levi-Civita connection, we also have

$$\begin{aligned}\xi(\alpha(X)) &= \xi\big(-\lambda^{-1} g(\xi, X)\big) = -\lambda^{-1}(g(\nabla_\xi \xi, X) + g(\xi, \nabla_\xi X)) \\ &= -\lambda^{-1} g(\xi, [\xi, X]) - \lambda^{-1}(g(\nabla_\xi \xi, X) + g(\xi, \nabla_X \xi)) \\ &= \alpha([\xi, X]),\end{aligned}$$

where we used that $\xi$ is a Killing vector field to conclude that $g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0$. But then we see that $i_\xi d\alpha = 0$, and $d\alpha$ is a basic 2-form on $M$. Hence, it is related to a two-form $K \in \Omega^2(S)$ such that $\pi^* K = d\alpha$.

The curvature $K$ is closed because $\pi^*(dK) = d(\pi^* K) = d(d\alpha) = 0$ and $\pi$ is a surjective smooth submersion so the pullback is injective. Hence, we can consider its equivalence class in the second de Rham cohomology $[K] \in H^2_{\mathrm{dR}}(S)$. This is a characteristic class, which is known to be independent of the connection. Since $\pi\colon M \to S$ is a trivial principal bundle, it can be given the trivial connection and then the curvature vanishes. But then the fact that the characteristic classes are independent of the connection tells us that $[K] = 0 \in H^2_{\mathrm{dR}}(S)$. In other words, $K$ is an exact one-form on $S$.

### Observer space and Cauchy surfaces

The observer space $S$ contains a lot of information about the topology of the spacelike part of $M$. Any edgeless, achronal, embedded spacelike hypersurface in $M$ is diffeomorphic to $S$ [40, Theorem 3]. In particular, if $M$ is globally hyperbolic, a smooth spacelike Cauchy surface is diffeomorphic to $S$. The smoothness of the Cauchy surface is not an extra condition on $(M, g)$ as any globally hyperbolic spacetime admits a smooth spacelike Cauchy surface [14]. It is easy to prove directly that the observer space is diffeomorphic to a smooth spacelike Cauchy surface:

**Proposition 2.5.** *Let $(M, g)$ be a globally hyperbolic, stationary spacetime with a smooth spacelike Cauchy surface $\Sigma$ and observer space $S$. Then $\Sigma$ and $S$ are diffeomorphic.*

*Proof.* A globally hyperbolic spacetime is, in particular, chronological so Theorem 2.3 tells us that $S$ is indeed a three-dimensional manifold. A Cauchy surface $\Sigma$ is embedded in $M$, so we can restrict $\pi$ to $\Sigma$. Let $\gamma \in S$, then it is a maximal integral curve of a complete timelike Killing vector field $\xi$. In particular, it is an inextendible timelike curve, so it intersects $\Sigma$ exactly once. Therefore, $\pi|_\Sigma\colon \Sigma \to S$ is a smooth bijection. Let $p \in \Sigma$ and suppose we have $v \in T_p\Sigma \subseteq T_pM$ such that $d\pi_p(v) = 0$. Then $v = a\xi_p$ for some $a \in \mathbb{R}$ because $\operatorname{Ker} d\pi_p = \mathbb{R}\xi_p$, but $v$ cannot be timelike as $v \in T_p\Sigma$. Therefore, we must have $a = 0$ and $v = 0$, so $d\pi_p|_{T_p\Sigma}\colon T_p\Sigma \to T_pS$ is injective. Therefore, $\pi|_\Sigma$ is a smooth immersion, and $\dim \Sigma = \dim S$ implies that $\pi$ is a local diffeomorphism. Bijectivity tells us that $\pi|_\Sigma$ is a diffeomorphism. □

### Tensor fields and metrics on the observer space

Up to now, we only know that $S$ is a three-dimensional smooth manifold, but we also want to endow it with a Riemannian metric. To do so, we need to understand the tensors on $S$. This is solved by the following proposition due to Geroch. It has the same spirit as the fact that basic differential forms on a principal bundle live on the base space.

**Proposition 2.6.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$ and with observer space $S$. There is a $C^\infty(S)$-module isomorphism between the set of tensor fields $T'$ on $S$ and the set of tensor fields $T$ on $M$ such that $\mathcal{L}_\xi T = 0$ and all possible contractions between $T$ and $\xi$ vanish. Moreover, the correspondence commutes with tensor products and contractions.*

*Proof.* See [43, Appendix] or Appendix A. □

Recall that we defined a scalar field $\lambda$ on $M$ as the norm

$$\lambda = -g(\xi, \xi). \tag{2.1}$$

It looks pretty simple but it is a very important object for defining multipole moments because the mass of the system is residing in $\lambda$. It satisfies $\mathcal{L}_\xi \lambda = 0$ because $\xi$ is a Killing vector field and commutes with itself, so Proposition 2.6 tells us that $\lambda$ can be seen as a scalar field on $S$. More precisely, there is a smooth function $\lambda' \in C^\infty(S)$ such that $\pi^*\lambda' = \lambda$.

Another important tensor field is the covariant 2-tensor field defined by

$$h = \lambda g + \xi^\flat \otimes \xi^\flat. \tag{2.2}$$

As the following proposition shows, it turns out to reduce to a Riemannian metric on $S$.

**Proposition 2.7.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$ and with observer space $S$. The covariant 2-tensor field $h$ on $M$ defined by (2.2) corresponds to a Riemannian metric on $h'$ on $S$ via Proposition 2.6.*

*Proof.* We already know that $\mathcal{L}_\xi \lambda = 0$, $\mathcal{L}_\xi(\xi^\flat) = 0$ and $\mathcal{L}_\xi g = 0$, so we also have $\mathcal{L}_\xi h = 0$. Moreover,

$$h(\xi, X) = \lambda g(\xi, X) + g(\xi, \xi) g(\xi, X) = 0,$$

for all $X \in \mathfrak{X}(M)$, so the contractions of $h$ with $\xi$ vanish. Therefore, $h$ can also be seen as a covariant 2-tensor field $h'$ on $S$ defined by $\pi^* h' = h$. Since $h$ is symmetric, $h'$ is also symmetric. We want to check that $h'$ is positive-definite. Let $x \in S$ and $\widetilde{v} \in T_x S$, then we can take $p \in M$ and $v \in T_p M$ such that $x = \pi(p)$ and $\widetilde{v} = d\pi_p(v)$. Then we have

$$h'_x(\widetilde{v}, \widetilde{v}) = \left(\pi^* h'\right)_p(v, v) = h_p(v, v) = \lambda(p) g_p(v, v) + g_p(\xi_p, v) g_p(\xi_p, v).$$

Extend $e_0 = \frac{1}{\sqrt{\lambda(p)}} \xi_p$ to a basis $(e_0, e_1, e_2, e_3)$ of $T_p M$ such that $g_p(e_\mu, e_\nu) = \eta_{\mu\nu}$ and expand $v = v^\mu e_\mu$. Then we get

$$h'_x(\widetilde{v}, \widetilde{v}) = \lambda(p) v^\mu v^\nu (\eta_{\mu\nu} + \eta_{0\mu}\eta_{0\nu}) = \lambda(p) v^i v^j \delta_{ij}.$$

Since $\lambda(p) > 0$, we see that this is non-negative. Suppose it is zero, then we must have $v^1 = v^2 = v^3 = 0$, so $v$ is proportional to $\xi_p$. But $d\pi_p(\xi_p) = 0$ because $\pi$ is constant along the integral curves of $\xi$, so this gives $\widetilde{v} = d\pi_p(v) = 0$. Therefore, $h'_x$ is indeed positive-definite and $h'$ is a Riemannian metric on $S$. □

The metric determined by equation (2.2) may not be the most natural one on $S$. We can rescale $h$ with a conformal factor, giving a new metric. There are three somewhat natural choices:

1. The metric $h'$ on $S$ satisfying $\pi^* h' = h$, where $h$ determined by equation (2.2);
2. The conformal metric $\lambda'^{-1} h'$;
3. The conformal metric $\lambda'^{-2} h'$.

In the proof of Proposition 2.6 in Appendix A, we used the second choice. For $v, w \in (\operatorname{Ker} d\pi_p)^\perp$, this metric satisfies

$$\left(\pi^*\left(\lambda'^{-1} h'\right)\right)_p(v, w) = \lambda(p)^{-1}\Big(\lambda(p) g_p(v, w) + \xi_p^\flat(v) \otimes \xi_p^\flat(w)\Big) = g_p(v, v),$$

because $\xi_p^\flat(v) = g_p(\xi_p, v) = 0$ for $v \in (\operatorname{Ker} d\pi_p)^\perp$. Therefore, this metric turns $\pi$ into a pseudo-Riemannian submersion from $(M, \pi)$ to $\left(S, \lambda'^{-1} h'\right)$, thus it is the most natural metric on $S$ when we view it is a quotient space of $M$.

The third suitable metric is $\lambda'^{-2} h'$, which has a nice property. If we assume that $(M, g)$ is globally hyperbolic, then the Riemannian manifold $\left(S, \lambda'^{-2} h'\right)$ is complete [40, Theorem 8]. We know that for any Riemannian manifold there is a conformally related metric turning it into a complete Riemannian manifold. In this case, the result already gives us such a conformally related metric. We repeat the result here with a small correction in the proof.[2]

**Proposition 2.8.** *Let $(M, g)$ be a globally hyperbolic, stationary spacetime, with the Riemannian manifold $\left(S, \lambda'^{-2} h'\right)$ as constructed above. Then $\left(S, \lambda'^{-2} h'\right)$ is a complete Riemannian manifold.*

*Proof.* By Theorem 2.3, the observer space $S$ is a smooth manifold and $\pi\colon M \to S$ a smooth surjective submersion. Above we already saw that $\widetilde{h} = \lambda'^{-2} h'$ is a Riemannian metric on $S$. We want to show that $\left(S, \widetilde{h}\right)$ is geodesically complete. Let $\gamma\colon [0, L) \to S$ be a unit-speed geodesic in $S$ with respect to $\widetilde{h}$, and take a point $p \in M$ such that $\pi(p) = \gamma(0)$. Then there is a unique smooth curve $\sigma\colon [0, L) \to M$ that horizontally lifts $\gamma$, meaning $\pi \circ \sigma = \gamma$ and $\sigma'(s) \in (\operatorname{Ker} d\pi_{\sigma(s)})^\perp$, because $\pi\colon M \to S$ is a principal bundle and we endowed it with a connection [82, Theorem 10.2].

Let $F\colon \mathbb{R} \times [0, L) \to M$ be given by $F(t, s) = t \cdot \sigma(s)$. Then

$$dF_{(t_0, s_0)}\left(\left.\frac{\partial}{\partial t}\right|_{(t_0, s_0)}\right) = \xi_{t_0 \cdot \sigma(s_0)}, \tag{2.3}$$

and

$$dF_{(t_0, s_0)}\left(\left.\frac{\partial}{\partial s}\right|_{(t_0, s_0)}\right) = d(\theta_{t_0})_{\sigma(s_0)}(\sigma'(s_0)). \tag{2.4}$$

We have $\sigma'(s_0) \in (\operatorname{Ker} d\pi_{\sigma(s_0)})^\perp$, which implies $d(\theta_{t_0})_{\sigma(s_0)}(\sigma'(s_0)) \in (\operatorname{Ker} d\pi_{t_0 \cdot \sigma(s_0)})^\perp$ by Proposition 2.4. The vectors (2.3) and (2.4) are nonzero because $\gamma'(s_0) \neq 0$, non-null and they

[2] In [40], it is assumed that $P = \bigcup_{t \in [0, L)} \pi^{-1}(\sigma(t))$ in the proof of Proposition 2.8 is a submanifold of $M$. It is not clear to me whether this is always true. In general, geodesics can intersect themselves as is the case on a cone for example. Luckily, we do not need that $P$ is a submanifold of $M$ in this proof.

are orthogonal, so they are linearly independent. This shows that $F$ is a smooth immersion. Moreover,

$$\begin{aligned}
F^*(\lambda^{-1}g)\left(\frac{\partial}{\partial t},\frac{\partial}{\partial t}\right)(t_0,s_0) &= (\lambda(t_0\cdot\sigma(s_0)))^{-1}g_{t_0\cdot\sigma(s_0)}(\xi_{t_0\cdot\sigma(s_0)},\xi_{t_0\cdot\sigma(s_0)}) = -1,\\
F^*(\lambda^{-1}g)\left(\frac{\partial}{\partial t},\frac{\partial}{\partial s}\right)(t_0,s_0) &= (\lambda(t_0\cdot\sigma(s_0)))^{-1}g_{t_0\cdot\sigma(s_0)}\big(\xi_{t_0\cdot\sigma(s_0)},d(\theta_{t_0})_{\sigma(s_0)}(\sigma'(s_0))\big) = 0,\\
F^*(\lambda^{-1}g)\left(\frac{\partial}{\partial s},\frac{\partial}{\partial s}\right)(t_0,s_0) &= (\lambda(t_0\cdot\sigma(s_0)))^{-1}g_{t_0\cdot\sigma(s_0)}\big(d(\theta_{t_0})_{\sigma(s_0)}(\sigma'(s_0)),d(\theta_{t_0})_{\sigma(s_0)}(\sigma'(s_0))\big)\\
&= (((\theta_{t_0})^*\lambda)(\sigma(s_0)))^{-1}((\theta_{t_0})^*g)_{\sigma(s_0)}\big(\sigma'(s_0),\sigma'(s_0)\big)\\
&= (\lambda(\sigma(s_0)))^{-1}g_{\sigma(s_0)}\big(\sigma'(s_0),\sigma'(s_0)\big).
\end{aligned}$$

Now, $d\pi_p(\sigma'(s_0)) = \gamma'(s_0)$ and the fact that $\gamma$ has unit speed gives $\widetilde{h}_{\gamma(s_0)}(\gamma'(s_0),\gamma'(s_0)) = 1$. Hence, using that $\sigma'(s_0)$ is orthogonal to $\xi_{\sigma(s_0)}$,

$$(\lambda(\sigma(s_0)))^{-1}g_{\sigma(s_0)}\big(\sigma'(s_0),\sigma'(s_0)\big) = \left(\pi^*\widetilde{h}\right)_{\sigma(s_0)}\big(\sigma'(s_0),\sigma'(s_0)\big) = \widetilde{h}_{\gamma(s_0)}(\gamma'(s_0),\gamma'(s_0)) = 1.$$

So, on $\mathbb{R}\times[0,-L)$ we find that $F^*(\lambda^{-1}g) = -dt^2+ds^2$. Let $x = (-2L)\cdot p$ and $y = (2L)\cdot p$. Then we have $x = F(-2L,0)$ and $y = F(2L,0)$. Let $t_0 \in [0,L)$, then $F(0,t) = p$ and we clearly have $x \ll p \ll y$. Suppose $t_0 > 0$ and define the curve $f\colon [0,1] \to \mathbb{R}\times[0,L)$ by $f(t) = (2L(t-1),t_0 t)$ and $f'(t) = (2L,t_0)$. Then $F\circ f$ is a curve in $M$ with $(F\circ f)'(t) = dF_{f(t)}(f'(t))$ and

$$(\lambda(F(f(t))))^{-1}g_{F(f(t))}((F\circ f)'(t),(F\circ f)'(t)) = (F^*(\lambda^{-1}g))_{f(t)}(f'(t),f'(t)) = -(2L)^2+t_0^2 < 0.$$

Since $\lambda$ is positive, this shows that $F\circ f$ is a timelike curve in $M$. Hence, $x \ll \sigma(t_0)$, and similarly we have $\sigma(t_0) \ll y$. Hence, $\sigma$ is a curve that lies entirely in the causal diamond $J(x,y) = J^+(x)\cap J^-(y)$. By global hyperbolicity of $M$, this is compact. But then there must be a point $q \in J(x,y)$ such that $\lim_{t\nearrow L}\sigma(t) = q$. Then we also have $\lim_{t\nearrow L}\gamma(t) = \pi(q)$, and hence $\gamma$ is extendible. This proves that $(S,\widetilde{h})$ is complete. □

It would be interesting to know whether Proposition 2.8 can be reversed. In general, that is not possible, but it can be done with some extra assumptions. First, we need to assume that the observer space is a manifold, for which it is sufficient to assume the spacetime $(M,g)$ is chronological or non-totally vicious as we saw above. We call $(M,g)$ future-distinguishing if $I^+(p) = I^+(q)$ implies $p = q$. Moreover, we call $(M,g)$ causally bounded if $\pi(I^+(p)\cap I^-(q))$ is bounded in $\big(S,\lambda'^{-2}h'\big)$ for every $p,q \in M$. If $(M,g)$ is globally hyperbolic, then it is easy to check that it is future-distinguishing and causally bounded. Conversely, if the observer space $S$ is a manifold such that $\pi$ is a surjective smooth submersion, and $(M,g)$ is future-distinguishing, causally bounded, and $\big(S,\lambda'^{-2}h'\big)$ is a complete Riemannian manifold, then $(M,g)$ must be globally hyperbolic [50, Theorem 2.15].

Even though the other options may seem more natural, we will stick to the metric $h'$ determined by equation (2.2). This metric is most useful when invoking the Einstein equations [32, 43] as we will see in Section 2.4. In this metric on $S$, the measuring instruments are scaled to agree with the interval between pulses of light emitted "at infinity" with a prearranged frequency [48]. For a stationary spacetime, we can locally take coordinates $(t,x^1,x^2,x^3)$ such that $\frac{\partial}{\partial t} = \xi$ and the coordinate representation of $\pi$ is the projection onto the last three components for some coordinates $(y^1, y^2, y^3)$ on $S$. Then the metric looks like

$$g = -\lambda(dt - \sigma_i dx^i)^2 + \lambda^{-1}\gamma_{ij}dx^i dx^j,$$

where $\lambda$, $\sigma_i$ and $\gamma_{ij}$ are smooth functions that are independent of $t$. In these coordinates, the tensor $h$ on $M$ looks like $h_{00} = h_{0i} = 0$, and

$$h_{ij} = \lambda(\lambda^{-1}\gamma_{ij} - \lambda\sigma_i\sigma_j) + \lambda^2\sigma_i\sigma_j = \gamma_{ij}.$$

So, the component functions of $h$ are independent of $t$, and $h'_{ij} = \gamma'_{ij}$, where $\gamma'_{ij}$ is determined by $\pi^*\gamma'_{ij} = \gamma_{ij}$. That is, $\gamma'_{ij}$ is the same as $\gamma_{ij}$ but we do not understand $t$ as a variable anymore. They serve as the component functions of the Riemannian metric on $S$.

## 2.3 Twist covector field

The goal of this section is to define a crucial ingredient for the multipole moments. The twist covector field contains a lot of information about the dynamics of the spacetime. The main result of this section is to calculate the exterior derivative of the twist one-form, and show that the twist one-form is closed in a vacuum solution of the Einstein equations. Remember that $(M, g)$ is a stationary spacetime with stationary vector field $\xi$ and observer space $S$.

### Definition of the twist covector field

**Definition 2.9.** Let $(M, g)$ be an orientable, stationary spacetime with stationary vector field $\xi$. Then the *twist covector field* $\omega \in \Omega^1(M)$ is defined by

$$\omega = -*\left(\xi^\flat \wedge d\xi^\flat\right), \tag{2.5}$$

where $*$ denotes the Hodge star operator. In abstract index notation, this reads

$$\omega_\mu = \varepsilon_{\mu\nu\rho\sigma}\xi^\nu\nabla^\rho\xi^\sigma.$$

Alternatively, we can also write

$$\omega = -i_\xi * d\xi^\flat.$$

The one-form $\omega$ is called the twist covector field of $\xi$. We have $\omega = 0$ if and only if $\xi^\flat \wedge d\xi^\flat = 0$, which holds if and only if $\operatorname{Ker}\xi^\flat = (\mathbb{R}\xi)^\perp$ is an involutive distribution on $M$. So, we see that $\omega = 0$ if and only if $\xi$ is a static vector field for $(M, g)$, where a static vector field is a stationary vector field whose orthogonal distribution is involutive. An involutive distribution corresponds to a foliation, and here it gives a foliation whose leaves are orthogonal to $\xi$. A spacetime $(M, g)$ admitting a static vector field $\xi$ is called static itself. The twist covector field measures the failure for a stationary vector field to be a static vector field.

It is clear that $i_\xi\omega = 0$ because $*d\xi^\flat$ is a two-form, so $i_\xi i_\xi * d\xi^\flat = 0$. Moreover, $\mathcal{L}_\xi g = 0$ and the divergence of a Killing vector field vanishes, from which we see that the Lie derivative with respect to $\xi$ commutes with the Hodge star operator. Moreover, $\mathcal{L}_\xi\xi = 0$ gives $\mathcal{L}_\xi\big(\xi^\flat\big) = (\mathcal{L}_\xi\xi)^\flat = 0$ because $\mathcal{L}_\xi$ commutes with raising and lowering. Then we also have $\mathcal{L}_\xi\big(d\xi^\flat\big) = d\mathcal{L}_\xi\big(\xi^\flat\big) = 0$, and we conclude that $\mathcal{L}_\xi\omega = 0$. Therefore, $\omega$ reduces to a covector field on $S$ by Proposition 2.6. More precisely, there exists $\omega' \in \Omega^2(S)$ such that $\pi^*\omega' = \omega$.

### Exterior derivative of the twist one-form

In a vacuum solution of the Einstein equations, the twist covector field turns out to be a closed one-form. The remaining part of this section is devoted to calculating the exterior derivative of the twist one-form. First, recall that the Riemann curvature tensor is a $(1,3)$-tensor field on $M$ defined by

$$(X,Y,Z) \mapsto R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Here, $\nabla$ denotes the Levi-Civita connection. We can also lower the last index, giving the $(0,4)$-tensor field $Rm$ defined by

$$Rm(X,Y,Z,W) = g(R(X,Y)Z,W).$$

Both tensors are called the Riemann curvature tensor. Contracting the first and last index gives the Ricci tensor

$$Rc = C_{14}(Rm).$$

We can express the exterior derivative of the twist one-form in terms of the Ricci tensor. To do this, we use the Kostant formula [66].

**Lemma 2.10** (Kostant formula)**.** *Let $(M,g)$ be a pseudo-Riemannian manifold with a Killing vector field $\xi$. Then*

$$\nabla^2_{X,Y}\xi = R(X,\xi)Y, \tag{2.6}$$

*for all $X, Y \in \mathfrak{X}(M)$, or, equivalently,*

$$\left(\nabla^2\xi\right)\left(Z^\flat, Y, X\right) = Rm(X,\xi,Y,Z), \tag{2.7}$$

*for all $X, Y, Z \in \mathfrak{X}(M)$.*

*Proof.* A proof in abstract index notation can be found in [112, p. 442]. We will prove it in the mathematical, global notation used in the statement of the lemma itself.

Let $X, Y, Z \in \mathfrak{X}(M)$. Since $\xi$ is a Killing vector field, we have $g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi) = 0$. Taking the derivative of this equation along $Z$ gives

$$g(\nabla_Z\nabla_X\xi, Y) + g(\nabla_X\xi, \nabla_Z Y) + g(\nabla_Z X, \nabla_Y\xi) + g(X, \nabla_Z\nabla_Y\xi) = 0,$$

where we used that the Levi-Civita connection is compatible with $g$. Cyclically permuting $X$, $Y$ and $Z$ also gives the equations

$$g(\nabla_Y\nabla_Z\xi, X) + g(\nabla_Z\xi, \nabla_Y X) + g(\nabla_Y Z, \nabla_X\xi) + g(Z, \nabla_Y\nabla_X\xi) = 0,$$

and

$$g(\nabla_X\nabla_Y\xi, Z) + g(\nabla_Y\xi, \nabla_X Z) + g(\nabla_X Y, \nabla_Z\xi) + g(Y, \nabla_X\nabla_Z\xi) = 0.$$

Adding the last two equations and subtracting the first one, gives

$$\begin{aligned} g(\nabla_Y\nabla_Z\xi - \nabla_Z\nabla_Y\xi, X) + g(\nabla_Y Z - \nabla_Z Y, \nabla_X\xi) + g(\nabla_X\nabla_Z\xi - \nabla_Z\nabla_X\xi, Y) \\ + g(\nabla_X Z - \nabla_Z X, \nabla_Y\xi) + g(\nabla_X\nabla_Y\xi + \nabla_Y\nabla_X\xi, Z) + g(\nabla_X Y + \nabla_Y X, \nabla_Z\xi) = 0. \end{aligned}$$

Using that the Levi-Civita connection is torsion-free and the vector field $\xi$ is a Killing vector field, we find

$$g(\nabla_Y Z - \nabla_Z Y, \nabla_X \xi) = g([Y, Z], \nabla_X \xi) = -g(\nabla_{[Y,Z]}\xi, X).$$

Moreover,

$$g(\nabla_X Y + \nabla_Y X, \nabla_Z \xi) = -g(\nabla_{\nabla_X Y}\xi + \nabla_{\nabla_Y X}\xi, Z).$$

Therefore, we find

$$Rm(Y, Z, \xi, X) + Rm(X, Z, \xi, Y) + g\big(\nabla^2_{X,Y}\xi + \nabla^2_{Y,X}\xi, Z\big) = 0.$$

Together with the symmetries of the Riemann curvature tensor, this gives

$$g\big(\nabla^2_{X,Y}\xi + \nabla^2_{Y,X}\xi, Z\big) = Rm(X, \xi, Y, Z) + Rm(Y, \xi, X, Z).$$

Since this holds for all $Z \in \mathfrak{X}(M)$, we also have

$$\nabla^2_{X,Y}\xi + \nabla^2_{Y,X}\xi = R(X, \xi)Y + R(Y, \xi)X.$$

The Ricci identity tells us that [73, Theorem 7.14]

$$\nabla^2_{X,Y}\xi - \nabla^2_{Y,X}\xi = R(X, Y)\xi.$$

Therefore, we have

$$2\nabla^2_{X,Y}\xi = R(X, \xi)Y + R(Y, \xi)X + R(X, Y)\xi = R(X, \xi)Y - R(\xi, X)Y = 2R(X, \xi)Y,$$

where we used the algebraic Bianchi identity. This proves the Kostant formula in the version of equation (2.6). Applying $Z^\flat$ gives equation (2.7). □

Eventually, we need an expression for the Laplace-Beltrami operator on $\xi$. We denote the Laplace-Beltrami operator on any tensor field by $\Box_g$ and we define it by

$$\Box_g T = \nabla^\mu \nabla_\mu T,$$

in abstract index notation. Since $g$ is a Lorentzian metric, the Laplace-Beltrami operator is a wave operator. For a Riemannian metric $h$, the induced Laplace-Beltrami operator is an elliptic partial differential operator, and we denote it by $\Delta_h$. To find $\Box_g\xi$, we want to contract $X$ and $Z$ in equation (2.6), giving the following corollary.

**Corollary 2.11.** *Let $(M, g)$ be a pseudo-Riemannian manifold with a Killing vector field $\xi$, then*

$$g(\Box_g\xi, W) = -Rc(\xi, W) \tag{2.8}$$

*for all $W \in \mathfrak{X}(M)$.*

*Proof.* By the symmetries of the Riemann curvature tensor, we can rewrite equation (2.7) as

$$(\nabla^2\xi)\Big(W^\flat, Z, X\Big) = Rm(X, \xi, Z, W) = -Rm(X, \xi, W, Z).$$

Contracting $X$ and $Z$ gives

$$g(\Box_g\xi, W) = (\Box_g\xi)\Big(W^\flat\Big) = -Rc(\xi, W),$$

proving equation (2.8). □

Using Lemma 2.10 and Corollary 2.11, we are able to find an expression for the exterior derivative of the twist one-form. In particular, we want the exterior derivative to vanish when $(M, g)$ solves the Einstein equations in vacuum. From Corollary 2.11, we already see that the Laplace-Beltrami operator on a Killing vector field vanishes in a solution of the Einstein equations in vacuum, because the Ricci tensor vanishes in that case.

**Theorem 2.12.** *Let $(M, g)$ be a stationary spacetime with timelike Killing vector field $\xi$. Let $\omega$ denote the twist covector field of $\xi$, then*

$$d\omega = 2i_\xi * Rc(\xi, \cdot) = -2 * \left(\xi^\flat \wedge Rc(\xi, \cdot)\right). \tag{2.9}$$

*Proof.* A proof in abstract index notation can be found in [112, p. 164], but we take a different approach via the Weitzenböck identity. We have

$$d\omega = -di_\xi * d\xi^\flat = -\mathcal{L}_\xi \left(*d\xi^\flat\right) + i_\xi d * d\xi^\flat.$$

Since $\xi$ is a Killing vector field, the Lie derivative along $\xi$ commutes with the Hodge star operator. Hence, for the first term we have

$$\mathcal{L}_\xi \left(*d\xi^\flat\right) = *\mathcal{L}_\xi \left(d\xi^\flat\right) = *d\mathcal{L}_\xi \left(\xi^\flat\right) = 0,$$

and we are left with

$$d\omega = i_\xi d * d\xi^\flat.$$

Since $d * d\xi^\flat$ is a 3-form on $M$, $M$ is 4-dimensional and $g$ has Lorentzian signature, we have $d * d\xi^\flat = * * d * d\xi^\flat$. Let $\Box_g^H = *d * d + d * d*$ denote the Hodge Laplacian on $(M, g)$, then

$$d\omega = i_\xi d * d\xi^\flat = i_\xi * *d * d\xi^\flat = i_\xi * \Box_g^H \xi^\flat - i_\xi * d * d * \xi^\flat.$$

We know the divergence of a Killing vector field vanishes, which gives $di_\xi \varepsilon = 0$. But then we also have $d * \xi^\flat = 0$, and we are left with

$$d\omega = i_\xi * \Box_g^H \xi^\flat.$$

Using the Weitzenböck identity [89, Theorem 9.4.1], we can relate the Laplace-Beltrami operator with the Hodge Laplacian via

$$\Box_g^H \xi^\flat = -\Box_g \xi^\flat + Rc(\xi, \cdot) = 2Rc(\xi, \cdot). \tag{2.10}$$

We used Corollary 2.11 in the last equality, combined with the fact that $\Box_g$ commutes with raising and lowering because the connection does. Substituting (2.10) in the expression for $d\omega$ proves (2.9). □

In a vacuum solution of the Einstein equations, the Ricci tensor vanishes, so we easily see from Theorem 2.12 that $d\omega = 0$. Since $\omega$ lives on $S$, there is a one-form $\omega' \in \Omega^1(S)$ such that $\pi^*\omega' = \omega$. Then we have

$$\pi^* d\omega' = d\pi^*\omega' = d\omega = 0.$$

Since $\pi$ is a surjective submersion, the pullback is injective and we must have $d\omega' = 0$. So, the twist covector field is also closed on $S$. Then we can locally always find a primitive function

$f'$ such that $\omega' = df'$. We call $f'$ the local twist potential. If $U$ is the domain of $f'$, then we can also define $f$ on $\pi^{-1}(U) \subseteq M$ by $f(p) = f'(\pi(p))$, and we easily see that $\omega = df$ on $\pi^{-1}(U)$. The smooth function $f$ is also called the twist potential. For the multipole moments, we are interested in what happens at infinity in the observer space. We can also take the twist potential around infinity, but then we first need to define asymptotic flatness in 3.

## 2.4 Einstein equations

In this section, we want to translate the Einstein equations to $S$ [13, 43]. Remember that $(M, g)$ is a stationary spacetime with stationary vector field $\xi$ and observer space $S$ and twist covector field $\omega$. At the end of Section 2.2, we saw three metrics on $S$. We said that we will use the one corresponding to $h$ given by (2.2), but in this section (and only in this section!) we also need

$$\widetilde{h} = \lambda^{-1} h = g + \lambda^{-1} \xi^\flat \otimes \xi^\flat.$$

We denote their counterparts on $S$ by $h'$ and $\widetilde{h}'$, respectively, so $\pi^* h' = h$ and $\pi^* \widetilde{h}' = \widetilde{h}$. The Levi-Civita connection on $(S, h')$ is denoted by $D$ and we write $\widetilde{D}$ for the Levi-Civita connection with respect to $\widetilde{h}'$. On $(M, g)$, the Riemann curvature tensor and the Ricci tensor are denoted by $Rm$ and $Rc$, respectively. On $S$, we write $Rm'$ and $Rc'$ when working with $h'$ and we write $\widetilde{Rm}$ and $\widetilde{Rc}$ when working with $\widetilde{h}'$. We start this section with a brief discussion on constructing stationary spacetimes from the observer space. After that, we derive some identities on $M$, which allows us to translate the Einstein equations to $S$.

### Constructing stationary spacetimes

Given a three-dimensional manifold with some data, it is possible to construct a stationary spacetime $(M, g)$ that is a solution of the Einstein equations in vacuum and such that the original space is the observer space. This is a result by Geroch [43]. The data must consist of a manifold $S$ with a Riemannian metric $\widetilde{h}'$, a positive scalar field $\lambda'$ and a closed covector field $\omega'$ (or its potential) such that

$$\operatorname{div}_{\widetilde{h}'} \omega' = \frac{3}{2} \lambda'^{-1} \omega' \big(\operatorname{grad}_{\widetilde{h}'} \lambda'\big), \tag{2.11a}$$

$$\Delta_{\widetilde{h}'} \lambda' = \frac{1}{2} \lambda'^{-1} \big|d\lambda'\big|^2_{\widetilde{h}'} - \lambda'^{-1} \big|\omega'\big|^2_{\widetilde{h}'}, \tag{2.11b}$$

$$\widetilde{Rc} = \frac{1}{2} \lambda'^{-2} \Big(\omega' \otimes \omega' - \big|\omega'\big|^2_{\widetilde{h}'} \widetilde{h}'\Big) + \frac{1}{2} \lambda'^{-1} D^2 \lambda - \frac{1}{4} \lambda'^{-2} d\lambda' \otimes d\lambda'. \tag{2.11c}$$

Here, $\operatorname{div}_{\widetilde{h}'}$ denotes the divergence with respect to $\widetilde{h}'$ and can alternatively be written as $\operatorname{div}_{\widetilde{h}'} \omega' = \widetilde{D}^i \omega'_i$. To construct the spacetime, take a chart $(x^0 = t, x^1, x^2, x^3)$ such that $\xi = \partial/\partial t$. It follows from the (2.11b) that $R_{00}$ vanishes, closedness of $\omega'$ implies that $R_{i0}$ vanishes and the (2.11c) tells us that $R_{ij}$ vanishes. We will not discuss more about the construction; more details can be found in [43]. A similar construction can be done in electrovacuum [32].

We can also take the opposite path (and that is what we want to do here). We start with a spacetime $(M, g)$, and then we calculate $\operatorname{div}_{\widetilde{h}'} \omega'$, $\Delta_{\widetilde{h}'} \lambda'$ and $\widetilde{Rc}$. Then we get

$$\operatorname{div}_{\widetilde{h}'} \omega' = \frac{3}{2} \lambda'^{-1} \omega' \big(\operatorname{grad}_{\widetilde{h}'} \lambda'\big), \tag{2.12a}$$

$$\Delta_{\widetilde{h}'}\lambda' = \frac{1}{2}\lambda'^{-1}\big|d\lambda'\big|^2_{\widetilde{h}'} - \lambda'^{-1}\big|\omega'\big|^2_{\widetilde{h}'} + 2\rho, \tag{2.12b}$$

$$\widetilde{Rc} = \frac{1}{2}\lambda'^{-2}\Big(\omega' \otimes \omega' - \big|\omega'\big|^2_{\widetilde{h}'}\widetilde{h}'\Big) + \frac{1}{2}\lambda'^{-1}D^2\lambda - \frac{1}{4}\lambda'^{-2}d\lambda' \otimes d\lambda' + \mathcal{R}, \tag{2.12c}$$

where $\rho$ is a function on $S$ determined by $\rho \circ \pi = Rc(\xi, \xi)$, and $\mathcal{R}$ is a covariant 2-tensor field on $S$ determined by $(\pi^*\mathcal{R})_{\mu\nu} = \widetilde{h}^\rho_\mu \widetilde{h}^\sigma_\nu R_{\rho\sigma}$. Here, $\widetilde{h}^\rho_\mu$ is given by

$$\widetilde{h}^\rho_\mu = g^{\nu\rho}\widetilde{h}_{\mu\nu} = \delta^\rho_\mu + \lambda^{-1}\xi^\rho\xi_\mu,$$

and $\widetilde{R}$ should be seen as the projection of the Ricci tensor on $M$ onto $S$. Comparing (2.11) and (2.12), we get indeed that $\rho = 0$ and $\mathcal{R} = 0$ as is needed in vacuum. These equations contain all information about the Ricci tensor together with equation (2.9). Expressing the Ricci tensor for $(M, g)$ in terms of the stress-energy tensor and its trace translates the Einstein equations to $S$. The remainder of this section serves to prove (2.12) and to perform a conformal transformation to the metric $h'$ instead of $\widetilde{h}'$. We do it in three steps. We start with calculating $\operatorname{div}_g \omega$ and $\Box_g\lambda$ on $M$. After that, we turn to $\big(S, \widetilde{h}'\big)$ and the last step is to replace $\widetilde{h}'$ by $h'$.

### Some identities on the spacetime

We start with deriving the divergence of $\omega$ and Laplace-Beltrami operator on $\lambda$ on $(M, g)$. From the fact that $\xi$ is a Killing vector field, we know that $\nabla\xi^\flat$ is antisymmetric and $d\xi^\flat = -2\nabla\xi^\flat$. From equation (2.5) we get, using that $\mathcal{L}_\xi(\xi^\flat) = 0$,

$$*\Big(\xi^\flat \wedge \omega\Big) = -i_\xi * \omega = i_\xi\Big(\xi^\flat \wedge d\xi^\flat\Big) = -\lambda d\xi^\flat - \xi^\flat \wedge i_\xi d\xi^\flat = -\lambda d\xi^\flat - \xi^\flat \wedge d\lambda.$$

Therefore,

$$\nabla\xi^\flat = -\frac{1}{2}d\xi^\flat = \frac{1}{2}\lambda^{-1}\Big(*\Big(\xi^\flat \wedge \omega\Big) + \xi^\flat \wedge d\lambda\Big). \tag{2.13}$$

This equation allows us to calculate $\operatorname{div}_g \omega$ and $\Box_g\lambda$.

**Lemma 2.13.** *Let $(M, g)$ be a stationary spacetime with a stationary vector field $\xi$ and let $\omega$ be the twist covector field of $\xi$. Then the divergence of $\omega$ is*

$$\operatorname{div}_g \omega = 2\lambda^{-1}\omega\big(\operatorname{grad}_g \lambda\big). \tag{2.14}$$

*Proof.* Since the Levi-Civita tensor is parallel with respect to the Levi-Civita connection, we see that $\nabla_X$ commutes with the Hodge star operator. This gives

$$\nabla_X\omega = - * \Big(\nabla_X\xi^\flat \wedge d\xi^\flat + \xi^\flat \wedge \nabla_X d\xi^\flat\Big). \tag{2.15}$$

For the second term we find

$$\nabla_X\Big(d\xi^\flat\Big)(Z, Y) = -\frac{1}{2}\Big(\nabla^2\xi^\flat\Big)(Z, Y, X) = Rm(X, \xi, Y, Z) = Rm(Y, Z, X, \xi),$$

using Lemma 2.10. The algebraic Bianchi identity implies that

$$\sum_{\sigma \in S_3} (\operatorname{sgn}\sigma) Rm\big(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}, \xi\big) = 0,$$

from which we find

$$C_{12}\Big((X,Y)\mapsto -\Big(*\Big(\xi^\flat\wedge\nabla_X d\xi^\flat\Big)\Big)(Y)\Big)=0.$$

For the divergence of $\omega$, we are only left with

$$\begin{aligned}\operatorname{div}_g\omega &= C_{12}\Big((X,Y)\mapsto -\Big(*\Big(\nabla_X\xi^\flat\wedge d\xi^\flat\Big)\Big)(Y)\Big)\\ &= C_{12}\Big((X,Y)\mapsto -2\Big(*\Big(i_X\Big(\nabla\xi^\flat\Big)\wedge\nabla\xi^\flat\Big)\Big)(Y)\Big)\\ &= C_{12}\Big((X,Y)\mapsto -\Big(*\Big(i_X\Big(\nabla\xi^\flat\wedge\nabla\xi^\flat\Big)\Big)\Big)(Y)\Big).\end{aligned}$$

Substitution of (2.13) yields

$$\nabla\xi^\flat\wedge\nabla\xi^\flat=\frac{1}{4}\lambda^{-2}*\Big(\xi^\flat\wedge\omega\Big)\wedge*\Big(\xi^\flat\wedge\omega\Big)+\frac{1}{2}\lambda^{-2}*\Big(\xi^\flat\wedge\omega\Big)\wedge\xi^\flat\wedge d\lambda.$$

For the first term, observe that

$$*\Big(\xi^\flat\wedge\omega\Big)\wedge*\Big(\xi^\flat\wedge\omega\Big)=\xi^\flat\wedge\omega\wedge**\Big(\xi^\flat\wedge\omega\Big)=-\xi^\flat\wedge\omega\wedge\xi^\flat\wedge\omega=0.$$

For the second term,

$$*\Big(\xi^\flat\wedge\omega\Big)\wedge\xi^\flat\wedge d\lambda=-\lambda\omega\big(\operatorname{grad}_g\lambda\big)\varepsilon,$$

where $\operatorname{grad}_g\lambda=(d\lambda)^\sharp$ and $\varepsilon$ is the pseudo-Riemannian volume form. Hence,

$$*(i_X\varepsilon)=**X^\flat=X^\flat,$$

and the trace of $(X,Y)\mapsto X^\flat(Y)=g(X,Y)$ equals 4. This gives equation (2.14). □

**Lemma 2.14.** *Let $(M,g)$ be a stationary spacetime with a stationary vector field $\xi$ and let $\lambda=-g(\xi,\xi)$. Then applying Laplace-Beltrami operator to $\lambda$ gives*

$$\Box_g\lambda=\lambda^{-1}|d\lambda|_g^2-\lambda^{-1}|\omega|_g^2+2Rc(\xi,\xi). \tag{2.16}$$

*Proof.* For the Laplacian of $\lambda$, we have

$$\Box_g\lambda=C_{12}\big((X,Y)\mapsto\nabla^2_{X,Y}\lambda\big),$$

where

$$\begin{aligned}\nabla^2_{X,Y}\lambda&=\nabla_X(\nabla_Y\lambda)-\nabla_{\nabla_XY}\lambda=-2\nabla_X(g(\nabla_Y\xi,\xi))+2g(\nabla_{\nabla_XY}\xi,\xi)\\ &=-2g\big(\nabla^2_{X,Y}\xi,\xi\big)-2g(\nabla_Y\xi,\nabla_X\xi).\end{aligned}$$

Using Corollary 2.11, taking the trace for the first term gives

$$C_{12}\big((X,Y)\mapsto-2g\big(\nabla^2_{X,Y}\xi,\xi\big)\big)=-2g(\Box_g\xi,\xi)=2Rc(\xi,\xi).$$

For the second term, we use equation (2.13) and observe that we get a full contraction of $\nabla\xi^\flat$ with itself. Since $\xi(\lambda)=d\lambda(\xi)=0$, $\omega(\xi)=0$, and contractions of $*\big(\xi^\flat\wedge\omega\big)$ with $\xi$ vanish, we get

$$C_{12}((X,Y)\mapsto-2g(\nabla_Y\xi,\nabla_X\xi))=\lambda^{-1}|d\lambda|_g^2-\lambda^{-1}|\omega|_g^2.$$

This gives equation (2.16). □

### Einstein equations on the observer space

The next step is to calculate the divergence of $\omega'$ and the Laplace-Beltrami operator on $\lambda'$ on $\big(S, \widetilde{h}'\big)$. Then we need to know how the Levi-Civita connections are related. Let $X$ be a smooth vector field on $S$, then by Proposition 2.6 there is a smooth vector field $\widetilde{X}$ on $M$ such that $d\pi_p\Big(\widetilde{X}_p\Big) = X_{\pi(p)}$ for all $p \in M$ that is orthogonal to $\xi$. Since $\pi$ is a pseudo-Riemannian submersion to $S$ with respect to $\lambda'^{-1}h'$, we have [86]

$$\widetilde{\widetilde{D}_X Y} = \Big(\nabla_{\widetilde{X}}\widetilde{Y}\Big)^H = \nabla_{\widetilde{X}}\widetilde{Y} + \lambda^{-1} g\Big(\xi, \nabla_{\widetilde{X}}\widetilde{Y}\Big)\xi = \nabla_{\widetilde{X}}\widetilde{Y} - \frac{1}{2}\Big[\widetilde{X}, \widetilde{Y}\Big]^V.$$

Here, the subscript $H$ means taking the horizontal part and $V$ the vertical part. Let $Z$ be a vector field on $M$, then we can decompose $Z$ as $Z^H + Z^V$ with $Z^V = -\lambda^{-1}g(\xi, Z)\xi$ and $Z^H = Z - Z^V$. Then $Z^H$ and $Z^V$ are also smooth vector fields on $M$, but $Z^V$ is vertical and

$$g\big(\xi, Z^H\big) = g(\xi, Z) + \lambda^{-1} g(\xi, Z) g(\xi, \xi) = g(\xi, Z) - g(\xi, Z) = 0,$$

so $Z^H$ is horizontal. Take coordinates $\big(t, x^1, x^2, x^3\big)$ for $M$ and $\big(y^1, y^2, y^3\big)$ for $S$ such that $\xi = \frac{\partial}{\partial t}$ and the coordinate representation of $\pi$ is the projection onto the last three components. Given a vector field $X = X^i \frac{\partial}{\partial y^i}$ on $S$, we have $\widetilde{X} = \widetilde{X}^0 \frac{\partial}{\partial t} + \big(X^i \circ \pi\big)\frac{\partial}{\partial x^i}$ on $M$ for some smooth function $\widetilde{X}^0$ to make sure $\widetilde{X}$ is orthogonal to $\xi$. In particular, $\widetilde{X}^0 = \lambda^{-1}\xi_i\big(X^i \circ \pi\big)$. Hence, we can write

$$\Big(\widetilde{D}_i Y^j\Big) \circ \pi = \lambda^{-1}\xi_i \nabla_\xi \widetilde{Y}^j + \nabla_i \widetilde{Y}^j = \big(\delta_i^\mu + \lambda^{-1}\xi_i\xi^\mu\big)\nabla_\mu \widetilde{Y}^j = \widetilde{h}_i^\mu \nabla_\mu \widetilde{Y}^j.$$

This allows us to calculate the divergence of $\omega'$ and the Laplace-Beltrami operator on $\lambda'$. Using the Levi-Civita connection above, we first do the calculation with respect to $\widetilde{h}'$, and then with respect to $h'$ using a conformal transformation as $h' = \lambda'\widetilde{h}'$.

**Lemma 2.15.** *Let $(M, g)$ be a stationary spacetime with observer space $S$ and twist covector field $\omega'$ on $S$, then we have*

$$\operatorname{div}_{h'} \omega' = 2\lambda'^{-1}\omega'\big(\operatorname{grad}_{h'} \lambda'\big), \tag{2.17}$$

*Proof.* We have

$$\begin{aligned}\big(\operatorname{div}_{\widetilde{h}'} \omega'\big) \circ \pi &= \Big(\widetilde{D}_i \omega'^i\Big) \circ \pi = \big(\delta_i^\mu + \lambda^{-1}\xi_i\xi^\mu\big)\nabla_\mu \omega^i = \nabla_i \omega^i + \lambda^{-1}\xi_i \nabla_\xi \omega^i \\ &= \nabla_\mu \omega^\mu + \lambda^{-1}\xi_\mu \nabla_\xi \omega^\mu = \operatorname{div}_g \omega + \lambda^{-1}(\nabla_\xi \omega)(\xi).\end{aligned}$$

Substituting $X = \xi$ in equation (2.15), we see the second term vanishes by the symmetries of the Riemann curvature tensor. So, we only need to take care of the first term and by the same calculation as before we have

$$\nabla_\xi \omega = - * \Big(\nabla_\xi \xi^\flat \wedge d\xi^\flat\Big) = - * \Big(i_\xi\Big(\nabla\xi^\flat \wedge \nabla\xi^\flat\Big)\Big) = \frac{1}{2}\lambda^{-1}\omega\big(\operatorname{grad}_g \lambda\big)\xi^\flat.$$

So, using (2.14), we have

$$\big(\operatorname{div}_{\widetilde{h}'} \omega'\big) \circ \pi = 2\lambda^{-1}\omega\big(\operatorname{grad}_g \lambda\big) - \frac{1}{2}\lambda^{-1}\omega\big(\operatorname{grad}_g \lambda\big) = \frac{3}{2}\lambda^{-1}\omega\big(\operatorname{grad}_g \lambda\big).$$

This also gives

$$\operatorname{div}_{\widetilde{h}'} \omega' = \frac{3}{2}\lambda'^{-1}\omega'\big(\operatorname{grad}_{\widetilde{h}'} \lambda'\big).$$

Observe that this is precisely (2.12a). However, we are working with $h'$ instead of $\widetilde{h}'$, so we want to use $D$ instead of $\widetilde{D}$. The conformal transformation gives

$$\operatorname{div}_{h'} \omega' = \lambda' \operatorname{div}_{\widetilde{h}} \omega' + \frac{1}{2}\lambda'^{-1}\omega'\big(\operatorname{grad}_{h'} \lambda'\big) = 2\lambda'^{-1}\omega'(\operatorname{grad}_{h'} \lambda'),$$

on $S$, where we note that the $\lambda'$ in the first term is taken into the gradient as we now raise $d\lambda'$ with the metric $h'$. □

**Lemma 2.16.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$, observer space $S$ and twist covector field $\omega$. Then we have*

$$\Delta_{h'}\lambda' = \lambda'^{-1}\big|d\lambda'\big|^2_{h'} - \lambda'^{-1}\big|\omega'\big|^2_{h'} + 2\lambda'^{-1}\rho, \tag{2.18}$$

*where $\rho \in C^\infty(S)$ is defined by $\pi^*\rho = Rc(\xi, \xi)$.*

*Proof.* First, we note that $\mathcal{L}_\xi Rc = 0$ because $\mathcal{L}_\xi g = 0$, so we see that $Rc(\xi, \xi)$ is indeed constant along the integral curves of $\xi$ and there exists a smooth function $\rho$ on $S$ such that $\pi^*\rho = Rc(\xi, \xi)$. Using that $\xi$ is a Killing vector field, we easily see that $\nabla_\xi \xi = \frac{1}{2} \operatorname{grad}_g \lambda$ on $M$, and this gives

$$\Big(\widetilde{\Delta}_{\widetilde{h}'}\lambda'\Big) \circ \pi = \Box_g\lambda - \frac{1}{2}\lambda^{-1}|d\lambda|^2_g = \frac{1}{2}\lambda^{-1}|d\lambda|^2_g - \lambda^{-1}|\omega|^2_g + 2Rc(\xi, \xi).$$

Therefore,

$$\widetilde{\Delta}_{\widetilde{h}'}\lambda' = \frac{1}{2}\lambda'^{-1}\big|d\lambda'\big|^2_{h'} - \lambda'^{-1}\big|\omega'\big|^2_{h'} + 2\rho,$$

which is (2.12b). Performing the conformal transformation gives

$$\Delta_{h'}\lambda' = \lambda'^{-1}\widetilde{\Delta}_{\widetilde{h}'}\lambda' + \frac{1}{2}\lambda'^{-1}\big|d\lambda'\big|^2_{h'} = \lambda'^{-1}\big|d\lambda'\big|^2_{h'} - \lambda'^{-1}\big|\omega'\big|_{h'} + 2\lambda'^{-1}\rho,$$

with respect to $h'$. □

Finally, we want to translate the Ricci tensor to $S$. By O'Neill's formula, we have

$$\begin{aligned}\widetilde{Rm}(X, Y, Z, W) \circ \pi = Rm\Big(\widetilde{X}, \widetilde{Y}, \widetilde{Z}, \widetilde{W}\Big) &- \frac{1}{2}g\bigg(\Big[\widetilde{X}, \widetilde{Y}\Big]^V, \Big[\widetilde{Z}, \widetilde{W}\Big]^V\bigg)\\ &- \frac{1}{4}g\bigg(\Big[\widetilde{X}, \widetilde{Z}\Big]^V, \Big[\widetilde{Y}, \widetilde{W}\Big]^V\bigg) + \frac{1}{4}g\bigg(\Big[\widetilde{X}, \widetilde{W}\Big]^V, \Big[\widetilde{Y}, \widetilde{Z}\Big]^V\bigg).\end{aligned}$$

This relates the Riemann curvature tensor via a pseudo-Riemannian submersion.

**Lemma 2.17.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$, observer space $S$ and twist covector field $\omega$. Let $\rho$ be defined as in Lemma 2.16, then*

$$Rc' = \mathcal{R} - \lambda'^{-2}h'\rho + \frac{1}{2}\lambda'^{-2}\big(\omega' \otimes \omega' + d\lambda' \otimes d\lambda'\big), \tag{2.19}$$

*where $\mathcal{R}$ is a covariant 2-tensor field on $S$ such that $(\pi^*\mathcal{R})_{\mu\nu} = \widetilde{h}^\rho_\mu \widetilde{h}^\sigma_\nu R_{\rho\sigma}$.*

*Proof.* In coordinates, we have

$$\begin{aligned}\left[\widetilde{\frac{\partial}{\partial y^i}}, \widetilde{\frac{\partial}{\partial y^j}}\right]^V &= \left[\lambda^{-1}\xi_i\frac{\partial}{\partial t} + \frac{\partial}{\partial x^i}, \lambda^{-1}\xi_j\frac{\partial}{\partial t} + \frac{\partial}{\partial x^j}\right]^V = \left(\frac{\partial(\lambda^{-1}\xi_j)}{\partial x^i} - \frac{\partial(\lambda^{-1}\xi_i)}{\partial x^j}\right)\xi \\ &= \left(\lambda^{-1}\left(\frac{\partial \xi_j}{\partial x^i} - \frac{\partial \xi_i}{\partial x^j}\right) + \lambda^{-2}\left(\xi_i\frac{\partial\lambda}{\partial x^j} - \xi_j\frac{\partial\lambda}{\partial x^i}\right)\right)\xi \\ &= \left(2\lambda^{-1}\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\mu\xi_\nu\right)\xi = \left(2\lambda^{-1}\widetilde{h}_{[i}^\mu\widetilde{h}_{j]}^\nu\nabla_\mu\xi_\nu\right)\xi,\end{aligned}$$

where $\widetilde{h}_\nu^\mu = \delta_\nu^\mu + \lambda^{-1}\xi^\mu\xi_\nu$ and $T_{[\mu\nu]} = \frac{1}{2}(T_{\mu\nu} - T_{\nu\mu})$. This shows that

$$\widetilde{R}_{ijkl}\circ\pi = \widetilde{h}_{[i}^\mu\widetilde{h}_{j]}^\nu\widetilde{h}_{[k}^\rho\widetilde{h}_{l]}^\sigma\big(R_{\mu\nu\rho\sigma} + 2\lambda^{-1}\nabla_\mu\xi_\nu\nabla_\rho\xi_\sigma + 2\lambda^{-1}\nabla_\mu\xi_\rho\nabla_\nu\xi_\sigma\big).$$

Hence, for the Ricci tensor on $S$ we have

$$\begin{aligned}\widetilde{R}_{ij}\circ\pi &= \left(\widetilde{h}'^{kl}\widetilde{R}_{kijl}\right)\circ\pi = \widetilde{h}^{kl}\widetilde{h}_k^\rho\widetilde{h}_i^\mu\widetilde{h}_{[j}^\nu\widetilde{h}_{l]}^\sigma\big(R_{\rho\mu\nu\sigma} + 2\lambda^{-1}\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma + 2\lambda^{-1}\nabla_\rho\xi_\nu\nabla_\mu\xi_\sigma\big) \\ &= \widetilde{h}^{\rho\sigma}\widetilde{h}_i^\mu\widetilde{h}_j^\nu R_{\rho\mu\nu\sigma} + \lambda^{-1}\widetilde{h}^{\rho\sigma}\widetilde{h}_i^\mu\widetilde{h}_j^\nu(2\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma + \nabla_\rho\xi_\nu\nabla_\mu\xi_\sigma - \nabla_\rho\xi_\sigma\nabla_\mu\xi_\nu) \\ &= \widetilde{h}_i^\mu\widetilde{h}_j^\nu R_{\mu\nu} + \lambda^{-1}\xi^\rho\xi^\sigma\widetilde{h}_i^\mu\widetilde{h}_j^\nu R_{\rho\mu\nu\sigma} + 3\lambda^{-1}\widetilde{h}^{\rho\sigma}\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma,\end{aligned}$$

where we used that $\nabla_\rho\xi_\sigma$ is antisymmetric and $\widetilde{h}^{\rho\sigma}$ is symmetric. Using that $\xi$ is a Killing vector field, we have

$$\xi^\sigma R_{\rho\mu\nu\sigma} = \nabla_\nu\nabla_\rho\xi_\mu.$$

Therefore,

$$\xi^\rho\xi^\sigma R_{\rho\mu\nu\sigma} = \frac{1}{2}\nabla_\nu\nabla_\mu\lambda - \nabla_\nu\xi^\rho\nabla_\rho\xi_\mu = \frac{1}{2}\nabla_\nu\nabla_\mu\lambda - g^{\rho\sigma}\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma.$$

Now,

$$\xi^\rho\xi^\sigma\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma = -\frac{1}{4}\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\mu\lambda\nabla_\nu\lambda = -\frac{1}{4}\nabla_i\lambda\nabla_j\lambda,$$

and using equation (2.13) we find

$$g^{\rho\sigma}\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\rho\xi_\mu\nabla_\nu\xi_\sigma = -\frac{1}{4}\lambda^{-1}\left(\widetilde{h}_{ij}\omega_\alpha\omega^\alpha - \omega_i\omega_j - \nabla_i\lambda\nabla_j\lambda\right).$$

Therefore,

$$\widetilde{R}_{ij}\circ\pi = \widetilde{h}_i^\mu\widetilde{h}_j^\nu R_{\mu\nu} + \frac{1}{2}\lambda^{-1}\widetilde{h}_i^\mu\widetilde{h}_j^\nu\nabla_\mu\nabla_\nu\lambda - \frac{1}{4}\lambda^{-2}\nabla_i\lambda\nabla_j\lambda + \frac{1}{2}\lambda^{-2}\left(\omega_i\omega_j - \widetilde{h}_{ij}|\omega|_g^2\right).$$

One can easily check that there is a covariant 2-tensor field $\mathcal{R}$ on $S$ corresponding to $\widetilde{h}_\rho^\mu\widetilde{h}_\sigma^\nu R_{\mu\nu}$ on $M$, and then

$$\widetilde{R}_{ij} = \mathcal{R}_{ij} + \frac{1}{2}\lambda'^{-1}\widetilde{D}_i\widetilde{D}_j\lambda' - \frac{1}{4}\lambda'^{-2}\widetilde{D}_i\lambda'\widetilde{D}_j\lambda' + \frac{1}{2}\lambda'^{-2}\left(\omega_i'\omega_j' - \widetilde{h}_{ij}'|\omega'|_{\widetilde{h}'}^2\right).$$

Here, we recognise (2.12c). However, we want to work with $D$ instead of $\widetilde{D}$. In that case,

$$\begin{aligned}R_{ij}' &= \widetilde{R}_{ij} - \frac{1}{2}\lambda'^{-1}D_iD_j\lambda' + \frac{1}{4}\lambda'^{-2}D_i\lambda'D_j\lambda' + \frac{1}{2}\lambda'^{-1}h_{ij}'\Delta_{h'}\lambda' - \frac{3}{4}\lambda'^{-2}h_{ij}'|d\lambda'|_{h'}^2 \\ &= \mathcal{R}_{ij} - \lambda'^{-2}h_{ij}'\rho + \frac{1}{2}\lambda'^{-2}\big(\omega_i'\omega_j' + D_i\lambda'D_j\lambda'\big).\end{aligned}$$

□

We can transform the Einstein equations into equations on $S$ using (2.9), (2.17), (2.18), and (2.19). They are the exterior derivative of $\omega$, the divergence of $\omega$, the Laplaci-Beltrami operator applied to $\lambda$ and the Ricci tensor on $S$. The advantage of working with $h'$ instead of $\widetilde{h}'$ can already be seen a little bit from (2.19), which contains one term less than (2.12c). We will continue the discussion using the potentials for the multipole moments in Section 4.1.

# Chapter 3

# Asymptotic Flatness in Stationary Spacetimes

Our spacetime $(M, g)$ is stationary and we want to work on the observer space $S$, which is constructed in Section 2.2. The space $S$ is a three-dimensional Riemannian manifold and we want it to be asymptotically flat. The goal of this chapter is to understand what it means for a three-dimensional Riemannian manifold to be asymptotically flat. There are multiple inequivalent definitions of asymptotic flatness [9, 28, 42, 71]. Generally, one can distinguish two main schools of defining asymptotic flatness, one via picking some nice coordinates and one via the geometric idea of compactifications. Regarding the latter, Geroch defined a geometric notion of asymptotic flatness in 1970 [42] as is discussed in Section 3.1. This notion requires a one-point completion and we investigate uniqueness of the one-point completion in Section 3.2. In Section 3.3, we compare Geroch's notion to a coordinate-based definition of asymptotic flatness, which leads to some loss of regularity.

## 3.1 Definitions of asymptotic flatness

Recall from Section 2.2 that the observer space of a stationary spacetime is a three-dimensional Riemannian manifold. A stationary spacetime $(M, g)$ is called asymptotically flat if $(S, h')$ is asymptotically flat. The goal of this section is to introduce two inequivalent notions of asymptotic flatness of three-dimensional Riemannian manifolds. Since we only work on the observer space $S$ in this section, we forget about the prime in the metric $h'$ and just denote it by $h$ as no confusion can arise with the tensor on $M$ defined by equation (2.2).

The first approach to asymptotic flatness requires a coordinate system and assumes that the components of the metric tensor are the same as for the Euclidean metric up to order $\frac{1}{r}$. We introduce the notion of asymptotic flatness of Lee [71, Definition 3.5]. In principle, asymptotic flatness can be defined for a space with multiple ends and then the multipole moments can be defined for every end. However, we restrict ourselves to manifolds with only one end, constituting only one set of multipole moments. This can be interpreted as if you walk infinitely far away in any direction, you will always walk towards the same infinity.

**Definition 3.1.** A three-dimensional Riemannian manifold $(S, h)$ is called *coordinate-wise asymptotically flat* if there exists a bounded set $K$ and a diffeomorphism $\varphi\colon S \setminus K \to \mathbb{R}^3 \setminus \overline{\mathbb{B}}^3$, where $\overline{\mathbb{B}}^3$ is the closed unit ball in $\mathbb{R}^3$, such that if we use $\varphi$ as a coordinate chart with coordinates $x^1, x^2, x^3$, then, in that coordinate chart,[3]

$$h_{ij} = \delta_{ij} + O^2\big(r(x)^{-q}\big),$$

for some $q > \frac{1}{2}$, where $r(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

Say we have $q = 1$ and we are given such a coordinate chart $\varphi\colon M\setminus K \to \mathbb{R}^3\setminus\overline{\mathbb{B}}^3$ with Cartesian coordinates $x^1, x^2, x^3$, then we can introduce a new coordinate chart $\psi\colon M \setminus K \to \mathbb{B}^3 \setminus \{0\}$ with coordinates $y^i = \frac{x^i}{r(x)^2}$. Here, $\mathbb{B}^3$ denotes the open unit ball in $\mathbb{R}^3$. What is at infinity in the first coordinate chart, is at zero in the second coordinate chart. It seems natural to add the point $y = 0$ as a completion at infinity. This led Geroch to define asymptotic flatness via a conformal completion [42], which was motivated by a similar definition for asymptotic flatness at null infinity due to Penrose [88].

**Definition 3.2.** A three-dimensional Riemannian manifold $(S, h)$ is called *asymptotically flat* if there exists a Riemannian manifold $\big(\widetilde{S}, \widetilde{h}\big)$ and a function $\Omega \in C^2\big(\widetilde{S}\big)$ such that:

(i) $S = \widetilde{S}\setminus\big\{i^0\big\}$ for a single point $i^0 \in \widetilde{S}$, and the inclusion $\iota\colon S \hookrightarrow \widetilde{S}$ is a smooth embedding;
(ii) $\iota^*\widetilde{h} = (\iota^*\Omega)^2 h$;
(iii) $\Omega(i^0) = 0$, $d\Omega_{i^0} = 0$ and $\widetilde{D}(d\Omega)|_{i^0} = 2\widetilde{h}_{i^0}$, where $\widetilde{D}$ denotes the Levi-Civita connection of $\big(\widetilde{S}, \widetilde{h}\big)$.[4]

*Remark.* Since $S$ is an open subspace of $\widetilde{S}$ according to (i) in Definition 3.2, we see that (ii) implies that $\Omega$ restricts to a smooth nonvanishing function on $S$. Moreover, in local coordinates around $i^0$, condition (iii) tells us that the gradient of $\Omega$ vanishes at $i^0$ and the Hessian matrix is positive-definite at $i^0$. Therefore, $\Omega$ attains an isolated local minimum at $i^0$. Hence, the function $\Omega$ is positive on $S$.

## 3.2 Uniqueness of the one-point conformal completion

The Riemannian manifold $\big(\widetilde{S}, \widetilde{h}\big)$ in Definition 3.2 can be seen as a one-point conformal completion of $(S, h)$. If we want to work on $\widetilde{S}$, it is important to know whether the Riemannian manifold $\big(\widetilde{S}, \widetilde{h}\big)$ is uniquely determined by the Riemannian manifold $(S, h)$. The multipole moments will be defined as tensors at $i^0$ and we do not want the multipole moments to depend on the chosen one-point extension. In this section, we prove a uniqueness result. We can replace part of the proof by a clever use of the conformal Laplacian, which we study the the second half of this section.

### A uniqueness result

In Definition 3.1, the bounded region $K$ can contain all kind of strange things, but we do not feel them at infinity. The idea of the next theorem is that we can also remove a subset such

[3]Here, $O^2\big(r(x)^{-1}\big)$ refers to an unspecified function in $C^2_{-q}$. We say that $f \in C^2_{-q}$ if there exists a constant $C > 0$ such that $|f| \le Cr(x)^{-q}$, $|\partial_i f| \le Cr(x)^{-q-1}$, and $|\partial_i\partial_j f| \le Cr(x)^{-q-2}$.

[4]Geroch only assumes that $\widetilde{D}\Big(\widetilde{D}\Omega\Big)|_{i^0}$ is proportional to $\widetilde{h}_{i^0}$. The factor 2 is due to Hansen [48].

that we perform a one-point compactification for the leftover. The subset helps to distinguish "neighborhoods" of a possible singularity and $i^0$. In [42], Geroch also "proved" that the one-point completion is unique where he does not introduce such a subset. However, his "proof" is incorrect and Theorem 3.3 is a correction of Geroch's result.[5]

**Theorem 3.3.** *Let $(S, h)$ be a three-dimensional Riemannian manifold and let $K \subseteq S$ be a closed subset. Suppose there is a homeomorphism $\varphi\colon S \setminus \operatorname{Int} K \to \mathbb{R}^3 \setminus \mathbb{B}^3$ which restricts to a diffeomorphism between $S \setminus K$ and $\mathbb{R}^3 \setminus \overline{\mathbb{B}}^3$. If there exists a Riemannian manifold $\big(\widetilde{S}, \widetilde{h}\big)$ with $\Omega \in C^2\big(\widetilde{S}\big)$ satisfying conditions (i)-(iii) in Definition 3.2 and such that $\widetilde{S} \setminus \operatorname{Int} K$ is compact, then it is unique up to conformal transformations with conformal factor* 1 *at* $i^0$.

*Proof.* We proceed with the proof in four steps. First, we start with uniqueness of $\big(\widetilde{S}, \widetilde{h}\big)$ up to homeomorphism, then we prove uniqueness up to diffeomorphism, and then we prove uniqueness up to conformal transformations. In other words, first we show uniqueness of the topology, then of the smooth manifold structure, and then of the metric up to a conformal factor. Finally, we show that the conformal factor must be 1 at $i^0$.

**Step 1: uniqueness of $\widetilde{S}$ up to homeomorphism.** Let $\widetilde{S}$ satisfy condition (i) in Definition 3.2 and such that $\widetilde{S} \setminus \operatorname{Int} K$ is compact. We characterise the topology on $\widetilde{S}$, fixing it uniquely, based on four claims. First, we want to fix the topology on the subspaces $\widetilde{S} \setminus K \subseteq \widetilde{S} \setminus \operatorname{Int} K \subseteq \widetilde{S}$. Then, we show that $S$ and $\widetilde{S} \setminus K$ form an open cover of $\widetilde{S}$. The topologies given on $S$ and $\widetilde{S} \setminus K$ then uniquely determine the topology of $\widetilde{S}$.

**Claim 1.** *A subset $V \subseteq \widetilde{S} \setminus K$ is open if and only if either $V \subseteq S \setminus K$ is open or $i^0 \in V$ and $\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus V$ is compact in $S \setminus \operatorname{Int} K$.*

*Proof of Claim 1.* First, we want to identify the open subsets of $\widetilde{S} \setminus \operatorname{Int} K$ in the same way. Since $S \setminus \operatorname{Int} K$ is a locally compact Hausdorff space, it has a unique one-point compactification up to homeomorphism [81, Theorem 29.1]. By assumption, $\widetilde{S} \setminus \operatorname{Int} K$ is a one-point compactification of $S \setminus \operatorname{Int} K$, so its topology is fixed. In particular, a subset $U \subseteq \widetilde{S} \setminus \operatorname{Int} K$ is open if and only if either $U \subseteq S \setminus \operatorname{Int} K$ is open or $i^0 \in U$ and $\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus U$ is compact in $S \setminus \operatorname{Int} K$.

"$\Longrightarrow$": Let $V \subseteq \widetilde{S} \setminus K$ be an open subset, then there exists an open subset $U \subseteq \widetilde{S} \setminus \operatorname{Int} K$ such that $V = U \cap \big(\widetilde{S} \setminus K\big)$. For $U$, there are two possibilities. Firstly, if $U \subseteq S \setminus \operatorname{Int} K$, then we see that $V \subseteq S \setminus K$ is an open open subset. Secondly, suppose $i^0 \in U$ and $\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus U$ is compact in $S \setminus \operatorname{Int} K$. Then we have $i^0 \in V$. Since $\partial K$ is homeomorphic to $\partial \mathbb{B}^3 = \mathbb{S}^2$, it is compact, and

$$V = U \cap \big(\widetilde{S} \setminus K\big) = U \setminus \partial K.$$

This shows that

$$\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus V = \big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus (U \setminus \partial K) = \Big(\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus U\Big) \cup \partial K$$

is compact.

"$\Longleftarrow$": There are two cases to consider. For the first case, let $V \subseteq S \setminus K$ be an open subset, then $V \subseteq S \setminus \operatorname{Int} K$ is also open. But then $V \subseteq \widetilde{S} \setminus \operatorname{Int} K$ is also open, from which we can conclude that $V \subseteq \widetilde{S} \setminus K$ is an open subset. For the second case, let $V \subseteq \widetilde{S} \setminus K$ be a subset

[5] In [42], Geroch defined a topology where the open neighborhoods of $i^0$ are the subsets $U \cup \{i^0\}$ of $\widetilde{S}$ where $U$ is an open subset of $S$ with compact boundary. However, this does not define a topology.

containing $i^0$ and such that $\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus V$ is compact in $S \setminus \operatorname{Int} K$. Then we have that $V \subseteq \widetilde{S} \setminus \operatorname{Int} K$ is open, which also gives that $V \subseteq \widetilde{S} \setminus K$ is open. ■

**Claim 2.** *The family $\{S, \widetilde{S} \setminus K\}$ of subsets of $\widetilde{S}$ is an open cover of $\widetilde{S}$.*

*Proof of Claim 2.* Since $K \subseteq S$, it is clear that $\widetilde{S} = S \cup \big(\widetilde{S} \setminus K\big)$. The singleton $\{i^0\}$ is closed in $\widetilde{S}$ because $\widetilde{S}$ is Hausdorff by assumption, so $S = \widetilde{S} \setminus \{i^0\}$ is an open subset of $\widetilde{S}$.

We are left to show that $\widetilde{S} \setminus K$ is an open subset of $\widetilde{S}$. Let $\overline{K}$ denote the closure of $K$ in $\widetilde{S}$, then we are done if $\overline{K} = K$. Since $S \setminus K$ is an open subset of $S$ and $S$ is open in $\widetilde{S}$, the set $S \setminus K$ is also open in $\widetilde{S}$. Therefore, $\overline{K} \subseteq K \cup \{i^0\}$. Let $U$ be a coordinate domain for $\widetilde{S}$ centered at $i^0$. Since $\partial K$ is compact in $S$, it is also compact in $\widetilde{S}$ and $U \setminus \partial K$ is open in $\widetilde{S}$. Let $V$ be the connected component of $U \setminus \partial K$ containing $i^0$. Then $V$ is homeomorphic to an open, connected subset of $\mathbb{R}^3$. Moreover, $W = V \setminus \{i^0\}$ is also an open, connected subset of $\widetilde{S}$. Hence, the set $W$ is open and connected in $S$, and does not intersect $\partial K$. Then $W \cap \operatorname{Int} K$ and $W \cap (S \setminus K)$ form a disjoint open cover of $W$, so by connectivity only one of them can be nonempty. Suppose $W \cap (S \setminus K) = \emptyset$, then $V \cap \big(\widetilde{S} \setminus K\big) = \{i^0\}$. By construction of the subspace topology, $V \cap \big(\widetilde{S} \setminus K\big) = \{i^0\}$ is open in $\widetilde{S} \setminus K$, so $S \setminus \operatorname{Int} K$ is compact by Claim 1. But $S \setminus \operatorname{Int} K$ is homeomorphic $\mathbb{R}^3 \setminus \mathbb{B}^3$, which is not compact, so we arrived at a contradiction. Therefore, we must have $W \cap (S \setminus K) \neq \emptyset$, implying that $W \cap \operatorname{Int} K = \emptyset$. Hence, $V$ is an open neighborhood of $i^0$ in $\widetilde{S}$ that does not intersect $K$. Therefore, $i^0 \notin \overline{K}$ and we achieve that $\overline{K} = K$. ■

**Claim 3.** *Let $\mathcal{T}$ be the topology of $S$ and let $\mathcal{T}_{i^0}$ be the collection of open neighborhoods of $i^0$ in $\widetilde{S} \setminus K$, then $\mathcal{T} \cup \mathcal{T}_{i^0}$ is a basis for a topology on $\widetilde{S}$.*

*Proof of Claim 3.* We have $S \in \mathcal{T}$ and $\widetilde{S} \setminus K \in \mathcal{T}_{i^0}$, and these open subsets of $\widetilde{S}$ cover $\widetilde{S}$ by Claim 2. Therefore, each point in $\widetilde{S}$ is contained in an element of $\mathcal{T} \cup \mathcal{T}_{i^0}$. By definition of a basis for a topology on $\widetilde{S}$, we are only left to show that for any $x \in U \cap V$ with $U, V \in \mathcal{T} \cup \mathcal{T}_{i^0}$, there exists a subset $W \in \mathcal{T} \cup \mathcal{T}_{i^0}$ such that $x \in W \subseteq U \cap V$ [81, Section 2.13]. In particular, it suffices to show that $\mathcal{T} \cup \mathcal{T}_{i^0}$ is closed under taking intersections.

There are a few cases to consider, depending on whether $U$ and $V$ belong to $\mathcal{T}$ or $\mathcal{T}_{i^0}$. If $U, V \in \mathcal{T}$, then $U \cap V \in \mathcal{T}$ because a topology is closed under taking intersections. If $U \in \mathcal{T}$ and $V \in \mathcal{T}_{i^0}$, we have $U \cap V = U \cap \big(V \setminus \{i^0\}\big)$. Since $\{i^0\}$ is closed in $\widetilde{S} \setminus K$, the set $V \setminus \{i^0\}$ must be open in $\widetilde{S} \setminus K$, but then $V \setminus \{i^0\}$ is open in $S \setminus K$ by Claim 1, so it is open in $S$. Therefore, $U \cap V = U \cap \big(V \setminus \{i^0\}\big) \in \mathcal{T}$. Finally, if $U, V \in \mathcal{T}_{i^0}$, we have $i^0 \in U \cap V$ and

$$\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus (U \cap V) = \Big(\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus U\Big) \cup \Big(\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus V\Big),$$

which is compact because as a union of two compact sets. Hence, $U \cap V \in \mathcal{T}_{i^0}$ by Claim 1. We conclude that $\mathcal{T} \cup \mathcal{T}_{i^0}$ is closed under all possible intersections and it is a basis for a topology on $\widetilde{S}$. ■

**Claim 4.** *The topology of $\widetilde{S}$ is the topology generated by $\mathcal{T} \cup \mathcal{T}_{i^0}$.*

*Proof of Claim 4.* Let $\widetilde{\mathcal{T}}$ be the topology of $\widetilde{S}$. The collection $\mathcal{T} \cup \mathcal{T}_{i^0}$ of subsets of $\widetilde{S}$ consists of subsets that are either open in $S$ or in $\widetilde{S} \setminus K$. Since $S$ and $\widetilde{S} \setminus K$ are open in $\widetilde{S}$ by Claim 2, these subsets must also be open in $\widetilde{S}$. Hence, $\mathcal{T} \cup \mathcal{T}_{i^0} \subseteq \widetilde{\mathcal{T}}$, from which we conclude that the topology generated by $\mathcal{T} \cup \mathcal{T}_{i^0}$ must be contained in $\widetilde{\mathcal{T}}$.

Conversely, let $U \in \widetilde{\mathcal{T}}$. If $i^0 \notin U$, then we have $U = U \cap S \in \mathcal{T}$. If $i^0 \in U$, then $U \cap S \in \mathcal{T}$ and $U \cap \big(\widetilde{S} \setminus K\big) \in \mathcal{T}_{i^0}$ because $S, \widetilde{S} \setminus K \subseteq \widetilde{S}$ are open. But we also have

$$U = (U \cap S) \cup \Big(U \cap \big(\widetilde{S} \setminus K\big)\Big),$$

so $U$ is contained in the topology generated by $\mathcal{T} \cup \mathcal{T}_{i^0}$. Hence, $\widetilde{\mathcal{T}}$ equals the topology generated by $\mathcal{T} \cup \mathcal{T}_{i^0}$. ■

Claim 1 fixes $\mathcal{T}_{i^0}$ as the collection of subsets $V \subseteq \widetilde{S} \setminus K$ such that $i^0 \in V$ and $\big(\widetilde{S} \setminus \operatorname{Int} K\big) \setminus V$ is compact in $S \setminus \operatorname{Int} K$. Since the topology on $S$ and the subset $K$ are given, it fixes both $\mathcal{T}$ and $\mathcal{T}_{i^0}$. Hence, the topology on $\widetilde{S}$ is fixed by Claim 4. This gives uniqueness of $\widetilde{S}$ up to homeomorphism.

**Step 2: uniqueness of $\widetilde{S}$ up to diffeomorphism.** Moise's theorem tells us that every 3-dimensional topological manifold admits, up to diffeomorphism, a unique smooth structure [79]. By Step 1 we have uniqueness of $\widetilde{S}$ up to homeomorphism, and thus uniqueness of $\widetilde{S}$ up to diffeomorphism follows immediately.

**Step 3: uniqueness of $\big(\widetilde{S}, \widetilde{h}\big)$ up to conformal transformations.** Assume that we have two metrics $\widetilde{h}_1$ and $\widetilde{h}_2$ on $\widetilde{S}$ with conformal factors $\Omega_1$ and $\Omega_1$, respectively, satisfying the conditions (i) and (ii) of Definition 3.2. On $S$, we have $h = \iota^*\Big(\Omega_1^{-2}\widetilde{h}_1\Big) = \iota^*\Big(\Omega_2^{-2}\widetilde{h}_2\Big)$. As already remarked below Definition 3.2, the functions $\Omega_1$ and $\Omega_2$ are smooth and nonvanishing on $S$. Then $\omega = \Omega_2/\Omega_1$ is a well-defined, smooth function on $S$. Moreover, $\widetilde{h}_2 = (\Omega_2/\Omega_1)^2\widetilde{h}_1 = \omega^2\widetilde{h}_1$ on $S$. It remains to extend this property to $\widetilde{S} = S \cup \{i^0\}$. Let $(E_1, E_2, E_3)$ be an orthonormal frame on an open neighborhood $U$ of $i^0$ with respect to $\widetilde{h}_1$. Then we have

$$\omega^2 = \omega^2\widetilde{h}_1(E_1, E_1) = \widetilde{h}_2(E_1, E_1),$$

on $U \setminus \{i^0\}$. The right-hand side is a smooth, (strictly) positive function on $U$, so $\omega^2$ also extends smoothly to $i^0$ with a positive value. Therefore, $\omega$ also extends to a smooth, nonvanishing function on $\widetilde{S}$. By continuity, we must have $\widetilde{h}_2 = \omega^2\widetilde{h}_1$ on all of $\widetilde{S}$, establishing uniqueness up to conformal transformations.

**Step 4: Uniqueness of the conformal factor at $i^0$.** Let us compare the two metrics and conformal factors in light of condition (iii) of Definition 3.2. By Step 3, we have $\Omega_2 = \omega\Omega_1$ for a smooth, nonvanishing function $\omega$ on $\widetilde{S}$. Let $\widetilde{D}_i$ denote the Levi-Civita connection with respect to $\widetilde{h}_i$, for $i = 1, 2$. Then the relation for the Levi-Civita connection between conformal metrics [73, Proposition 7.29] gives

$$\widetilde{D}_2(d\Omega_2) = \widetilde{D}_1(d\Omega_2) - \omega^{-1}(d\Omega_2 \otimes d\omega + d\omega \otimes d\Omega_2) + \omega^{-1}d\Omega_2\Big(\operatorname{grad}_{\widetilde{h}_1}\omega\Big)\widetilde{h}_1.$$

When evaluating at $i^0$, the last three terms vanish because $d\Omega_2|_{i^0} = 0$ by condition (iii) in Definition 3.2. The first term is

$$\widetilde{D}_1(d\Omega_2) = \widetilde{D}_1(d(\omega\Omega_1)) = \widetilde{D}_1(\omega d\Omega_1 + \Omega_1 d\omega) = \omega\widetilde{D}_1(d\Omega_1) + d\omega \otimes d\Omega_1 + d\Omega_1 \otimes d\omega + \Omega_1\widetilde{D}_1(d\omega),$$

of which the last three terms also vanish at $i^0$ because of condition (iii) in Definition 3.2. So,

$$\widetilde{D}_2(d\Omega_2)\Big|_{i^0} = \omega(i^0)\,\widetilde{D}_1(\omega d\Omega_1)\Big|_{i^0},$$

and applying condition (iii) in Definition 3.2 once more yields

$$2\big(\omega\big(i^0\big)\big)^2\widetilde{h}_1\Big|_{i^0}=2\widetilde{h}_2\Big|_{i^0}=\widetilde{D}_2(d\Omega_2)\Big|_{i^0}=\omega\big(i^0\big)\,\widetilde{D}_1(d\Omega_1)\Big|_{i^0}=2\omega\big(i^0\big)\,\widetilde{h}_1\Big|_{i^0}.$$

Since $\omega$ is nonvanishing on $\widetilde{S}$, this is only possible if $\omega\big(i^0\big)=1$, fixing the conformal factor at $i^0$. □

We typically assume that the subset $K$ in Theorem 3.3 is bounded. To define multipole moments, we are only interested in what happens around $i^0$. Therefore, we can simply remove $K$ from the space $S$ and work with $S\setminus K$. Then, $K$ only serves to distinguish the "boundary" $\partial K\cong\partial\mathbb{B}^3$ from $i^0$.

### Alternative approach via the conformal Laplacian

In Step 2 of the proof above, we made use of Moise's theorem, which relies on algebraic topology. It is also dependent on the dimension. It is also possible to show uniqueness up to diffeomorphism directly, which is the approach originally taken by Geroch [42, Appendix]. This method relies more on analysis and geometry, and can be generalised to arbitrary dimensions. The idea is to use the fact that the smooth structure is completely determined by the space of smooth functions [84, Section 1.1]. Since $\widetilde{S}$ contains $S$ as an open subspace, the smooth structure of $\widetilde{S}$ restricted to $S$ is fixed, but we need a characterization for the smooth functions on a neighborhood of $i^0$. We prove such a characterisation in Theorem 3.6, but first we need a proposition and a lemma.

The idea is to construct the smooth functions based on solutions of the conformal Laplace equation. For three-dimensional manifolds $(S,h)$, the conformal Laplacian is $\Delta_h-\frac{1}{8}R$, where $\Delta_h$ denotes the Laplace-Beltrami operator with respect to $h$ and $R$ is the Ricci scalar. Generalising to arbitrary dimension $n$, we have the following result:

**Proposition 3.4.** *Let $(S,h)$ be a Riemannian manifold of dimension $n\geq 2$ and let $\widetilde{h}=\Omega^2h$ for some smooth, positive function $\Omega$ on $S$. Let $R$ and $\widetilde{R}$ denote the Ricci scalars with respect to $h$ and $\widetilde{h}$, respectively, and define $\widetilde{\varphi}\in C^\infty(S)$ by*

$$\widetilde{\varphi}=\Omega^{-\frac{n-2}{2}}\varphi.$$

*Then,*

$$\left(\Delta_{\widetilde{h}}-\frac{n-2}{4(n-1)}\widetilde{R}\right)\widetilde{\varphi}=\Omega^{-\frac{n+2}{2}}\left(\Delta_h-\frac{n-2}{4(n-1)}R\right)\varphi.$$

*Proof.* Let $D$ and $\widetilde{D}$ denote the Levi-Civita connections with respect to $h$ and $\widetilde{h}$, respectively. After performing the conformal transformation to the Levi-Civita connection, the second covariant derivative of $\widetilde{\varphi}$ becomes

$$\widetilde{D}\Big(\widetilde{D}\widetilde{\varphi}\Big)=\widetilde{D}(d\widetilde{\varphi})=D(d\widetilde{\varphi})-\Omega^{-1}(d\widetilde{\varphi}\otimes d\Omega+d\Omega\otimes d\widetilde{\varphi}-d\widetilde{\varphi}(\mathrm{grad}_h\,\Omega)h). \tag{3.1}$$

For the differential of $\widetilde{\varphi}$, we have

$$d\widetilde{\varphi}=-\frac{n-2}{2}\Omega^{-\frac{n}{2}}\varphi d\Omega+\Omega^{-\frac{n-2}{2}}d\varphi.$$

Then the second covariant derivative of $\widetilde{\varphi}$ with respect to $h$ becomes

$$
\begin{aligned}
D(d\widetilde{\varphi}) = {} & \frac{n(n-2)}{4}\Omega^{-\frac{n+2}{2}}\varphi d\Omega \otimes d\Omega - \frac{n-2}{2}\Omega^{-\frac{n}{2}}(d\varphi \otimes d\Omega + d\Omega \otimes d\varphi) - \frac{n-2}{2}\Omega^{-\frac{n}{2}}\varphi D(d\Omega) \\
& + \Omega^{-\frac{n-2}{2}} D(d\varphi).
\end{aligned}
$$

For the Laplace-Beltrami operator with respect to $\widetilde{h}$, we want to take the trace of equation (3.1) with respect to $\widetilde{h}$. Let $(E_1, \ldots, E_n)$ be a local orthonormal frame with respect to $h$, then $\left(\widetilde{E}_1, \ldots, \widetilde{E}_n\right)$ with $\widetilde{E}_i = \Omega^{-1}E_i$ for $i = 1, \ldots, n$ is an orthonormal frame with respect to $\widetilde{h}$. The equations above give

$$
\begin{aligned}
\Delta_{\widetilde{h}}\widetilde{\varphi} &= \sum_{i=1}^{n} \widetilde{D}\left(\widetilde{D}\widetilde{\varphi}\right)\left(\widetilde{E}_i, \widetilde{E}_i\right) = \Omega^{-2}\sum_{i=1}^{n} \widetilde{D}\left(\widetilde{D}\widetilde{\varphi}\right)(E_i, E_i) \\
&= \Omega^{-\frac{n+2}{2}} \sum_{i=1}^{n} D(D\varphi)(E_i, E_i) - \frac{n-2}{2}\Omega^{-\frac{n+4}{2}}\varphi \sum_{i=1}^{n} D(D\Omega)(E_i, E_i) \\
&\quad - (n-2)\Omega^{-\frac{n+4}{2}} \sum_{i=1}^{n} d\Omega(E_i)d\varphi(E_i) + \frac{n(n-2)}{4}\Omega^{-\frac{n+6}{2}}\varphi \sum_{i=1}^{n} d\Omega(E_i)d\Omega(E_i) \\
&\quad + (n-2)\Omega^{-\frac{n+6}{2}}\varphi \sum_{i=1}^{n} d\Omega(E_i)d\Omega(E_i) - 2\Omega^{-\frac{n+4}{2}} \sum_{i=1}^{n} d\Omega(E_i)d\varphi(E_i) \\
&\quad - \frac{n-2}{2}\Omega^{-\frac{n+6}{2}}\varphi d\Omega(\operatorname{grad}_h \Omega) \sum_{i=1}^{n} h(E_i, E_i) + \Omega^{-\frac{n+4}{2}} d\varphi(\operatorname{grad}_h \Omega) \sum_{i=1}^{n} h(E_i, E_i) \\
&= \Omega^{-\frac{n+2}{2}}\left(\Delta_h\varphi - \frac{n-2}{2}\Omega^{-1}\varphi\Delta_h\Omega - \frac{(n-2)(n-4)}{4}\Omega^{-2}\varphi d\Omega(\operatorname{grad}_h \Omega)\right).
\end{aligned}
$$

The Ricci scalar transforms as

$$
\widetilde{R} = \Omega^{-2}\big(R - 2(n-1)\Omega^{-1}\Delta_h\Omega - (n-1)(n-4)\Omega^{-2}d\Omega(\operatorname{grad}_h \Omega)\big).
$$

Combining both expressions, we find

$$
\left(\Delta_{\widetilde{h}} - \frac{n-2}{4(n-1)}\widetilde{R}\right)\widetilde{\varphi} = \Omega^{-\frac{n+2}{2}}\left(\Delta_h\varphi - \frac{n-2}{4(n-1)}R\right)\varphi. \qquad \square
$$

We want to use Proposition 3.4 to identify smooth functions on $\widetilde{S}$ with smooth functions on $S$. The following lemma provides one step of the correspondence.

**Lemma 3.5.** *Let $(S, h)$, $\left(\widetilde{S}, \widetilde{h}\right)$ and $\Omega$ be as in Definition 3.2. Let $\varphi$ be a continuous function on $\widetilde{S}$ that vanishes at $i^0$, is smooth on $S$ and solves*

$$
\left(\Delta_h - \frac{1}{8}R\right)\varphi = 0, \tag{3.2}
$$

*on $S$. Then the function $\widetilde{\varphi} = \Omega^{-\frac{1}{2}}\varphi$ defined on $S$ extends to a smooth function on $\widetilde{S}$.*

*Proof.* By Proposition 3.4, we have

$$\left(\widetilde{\Delta}_{\widetilde{h}} - \frac{1}{8}\widetilde{R}\right)\widetilde{\varphi} = 0, \tag{3.3}$$

on $S$, where $\widetilde{\varphi} = \Omega^{-\frac{1}{2}}\varphi$. Here, $\widetilde{\Delta}_{\widetilde{h}}$ denotes the Laplace-Beltrami operator on $\left(\widetilde{S}, \widetilde{h}\right)$, but we restrict it to $S$. It is not clear whether $\widetilde{\varphi}$ even extends continuously to a function on $\widetilde{S}$. Let $\left(U, \left(y^1, y^2, y^3\right)\right)$ be a normal coordinate chart for $\left(\widetilde{S}, \widetilde{h}\right)$ centered at $i^0$. Then we have $\widetilde{h}_{ij}\left(i^0\right) = \delta_{ij}$, and by Taylor's theorem

$$\Omega(y) = (\delta_{ij} + f_{ij}(y))y^i y^j,$$

for some continuous functions $f_{ij}$ with $f_{ij} = f_{ji}$ and $f_{ij}(0) = 0$. After possibly shrinking $U$, we can assume that the eigenvalues of the matrix with entries $\delta_{ij} + f_{ij}(y)$ are between $\frac{1}{4}$ and 4. Then we have $\frac{1}{4}r^2 \leq \Omega(y) \leq 4r^2$ where $r^2 = \delta_{ij}y^i y^j$. Since $\varphi$ is continuous, we can assume it is bounded on $U$ after shrinking $U$ even more if necessary. But then $\widetilde{\varphi} = \Omega^{-\frac{1}{2}}\varphi$ is bounded by a multiple of $\frac{1}{r}$. By a singularity theorem of Serrin [100, Theorem 6], the function $\widetilde{\varphi}$ must have a removable singularity at 0, or there exist positive constants $c_1 < c_2$ such that $\frac{c_1}{r} \leq |\widetilde{\varphi}| \leq \frac{c_2}{r}$. In the latter case, we have $\frac{c_1}{2} \leq |\varphi| \leq 2c_2$ on $U \setminus \left\{i^0\right\}$, but that contradicts continuity of $\varphi$ at $i^0$ with $\varphi\left(i^0\right) = 0$. Hence, $\widetilde{\varphi}$ has a removable singularity at 0. In particular, $\widetilde{\varphi}$ is bounded on a small enough neighborhood of 0 and $\widetilde{\varphi}$ is twice continuously differentiable and a solution of equation (3.3) by Bochner's theorem [19]. But then $\widetilde{\varphi}$ is smooth because the elliptic partial differential equation has smooth coefficients. □

With the results above, we are able to prove the following characterization of smooth functions on $\widetilde{S}$.

**Theorem 3.6.** *Let $(S, h)$, $\left(\widetilde{S}, \widetilde{h}\right)$ and $\Omega$ be as in Definition 3.2. Then a function $f \in C\left(\widetilde{S}\right)$ is smooth if and only if the restriction $f|_S$ is a smooth function on $S$ and there is a neighborhood $U$ of $i^0$ such that $f|_U = F \circ (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_k)$ for some smooth function $F\colon \mathbb{R}^k \to \mathbb{R}$ and some continuous functions $\varphi_i$ on $\widetilde{S}$ that vanish at $i^0$, are smooth on $S$, and satisfy* (3.2) *and $\widetilde{\varphi}_i = \Omega^{-\frac{1}{2}}\varphi_i$, for $i = 1, \ldots, k$.*

*Proof.* " $\Longleftarrow$ ": Since $f$ restricts to a smooth function on $S$, we are only left to show that $f$ is smooth on a neighborhood of $i^0$. By Lemma 3.5, the functions $\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_k$ are smooth on $\widetilde{S}$, but then $f|_U = F \circ (\widetilde{\varphi}_1, \ldots, \widetilde{\varphi}_k)$ is a smooth function on $U$. Therefore, $f$ is a smooth function on $\widetilde{S}$.

" $\Longrightarrow$ ": Let $f$ be a smooth function on $\widetilde{S}$, then $f$ restricts to a smooth function on $S$ because $S$ is a smooth submanifold of $\widetilde{S}$. Let $\left(y^1, y^2, y^3\right)$ be smooth coordinates centered at $i^0$. In this chart, equation (3.3) is an elliptic partial differential equation with smooth coefficients. For each $i = 1, 2, 3$, there exists a solution $\widetilde{x}^i$ of

$$\left(\widetilde{\Delta}_{\widetilde{h}} - \frac{1}{8}\widetilde{R}\right)\widetilde{x}^i = 0, \tag{3.4}$$

such that $\widetilde{x}^i$ has Hölder continuous derivatives of order 2 and such that $\widetilde{x}^i = 0$ and $\frac{\partial \widetilde{x}^i}{\partial y^j} = \delta^i_j$ at $y = 0$ [15, Theorem II.5.4.1]. Since the elliptic partial differential operator has smooth coefficients, we see that the functions $\widetilde{x}^i$ are smooth. Then, $F = \left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$ is a smooth map on a neighborhood of $i^0$ with $\frac{\partial \widetilde{x}^i}{\partial y^j}\Big|_{y=0} = \delta^i_j$. In particular, $dF|_{i^0}$ is invertible and $F$ restricts to a diffeomorphism between a neighborhood of $i^0 \in \widetilde{S}$ and a neighborhood $0 \in \mathbb{R}^3$ by the inverse function theorem. This gives a new smooth chart centered at $i^0$ on a possibly smaller neighborhood $U$, and the coordinates satisfy equation (3.4).

Define $x^i = \Omega^{\frac{1}{2}} \widetilde{x}^i$, then $x^i$ is a continuous function on $U$ that vanishes at $i^0$ and restricts to a smooth function on $U \setminus \{i^0\}$. By Proposition 3.4, we have

$$\left(\Delta_h - \frac{1}{8} R\right) x^i = 0, \tag{3.5}$$

on $U \setminus \{i^0\}$. Taking $\varphi_i = x^i$, we have $\widetilde{\varphi}_i = \widetilde{x}^i$ and since $\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$ are smooth coordinates for $\widetilde{S}$ around $i^0$, any smooth function can be represented as a smooth function of $\left(\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3\right)$. □

The theorem above is Geroch's characterization of smooth functions on $\widetilde{S}$. Given a topology on $\widetilde{S}$, it can be used to define a smooth structure on $\widetilde{S}$.

## 3.3 Comparison of different approaches to asymptotic flatness

Even though we motivated Definition 3.2 via Definition 3.1, the latter does not imply the former. The main issues lie in the regularity. The point $i^0$ from Definition 3.2 can always be added continuously in the way suggested above Definition 3.2, but it cannot be done smoothly in general. In this section we want to grasp the idea of how they relate to each other, for which we define a new type of regularity. With this new regularity class, we show that coordinate-wise asymptotic flatness implies asymptotic flatness and the conformal completion is unique in this weaker sense.

### Weakening the regularity

Let $(S, h)$ be a three-dimensional Riemannian manifold that is coordinate-wise asymptotically flat. Let $K \subseteq S$ be bounded and let $\varphi\colon M \setminus K \to \mathbb{R}^3 \setminus \overline{\mathbb{B}}^3$ be a diffeomorphism with induced coordinates $\left(x^1, x^2, x^3\right)$ such that

$$h_{ij}(x) = \delta_{ij} + O^2\left(r(x)^{-1}\right).$$

Here, $r(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Let $\psi\colon M \setminus K \to \mathbb{B}^3 \setminus \{0\}$ be given by $\psi(p) = \frac{\varphi(p)}{|\varphi(p)|^2}$, then $\psi$ is also a diffeomorphism and it induces coordinates $\left(y^1, y^2, y^3\right)$ with $y^i = \frac{x^i}{r(x)^2}$. We do not consider $r$ to be a function on the manifold, but on the coordinates. Therefore, $r(y) = r(x)^{-1}$. In this new coordinate system, we have

$$h_{ij}(y) = r(y)^{-4}\left(\delta_{ij} + O^2(r(y))\right).$$

It seems natural to pick $\Omega(p) = r(\psi(p))^2$ and then

$$\widetilde{h}_{ij}(y) = \delta_{ij} + O^2(r(y)).$$

In this way, the conformal metric $\widetilde{h}$ can be extended continuously to $y = 0$, but its regularity may be spoiled by the term $r(y)$. The Euclidean norm, however, is Lipschitz continuous, suggesting we should relax the transition functions between charts to be Lipschitz continuous at $i^0$ $(y = 0)$ rather than differentiable. This leads to an new regularity structure on $\widetilde{S}$, which we will now define following Chruściel [28].

**Definition 3.7.** For $k \in \mathbb{N}$ and $\alpha \in (0, 1]$, define the set of functions $A_{k,\alpha}(B_R(0))$ on the open ball $B_R(0) \subseteq \mathbb{R}^3$ of radius $R$ centered at the origin as the functions $f \in C^1(B_R(0))$ satisfying $f|_{B_R(0)\setminus\{0\}} \in C^k(B_R(0) \setminus \{0\})$ and

$$\left|\frac{\partial f}{\partial y^i}(y) - \frac{\partial f}{\partial y^i}(0)\right| \leq Cr(y)^\alpha, \left|\frac{\partial^2 f}{\partial y^{i_1}\partial y^{i_2}}(y)\right| \leq Cr(y)^{\alpha-1}, \ldots, \left|\frac{\partial^k f}{\partial y^{i_1} \ldots \partial y^{i_k}}(y)\right| \leq Cr^{\alpha-k+1},$$

for some constant $C$. Here, $\left(y^1, y^2, y^3\right)$ are Cartesian coordinates on $B_R(0)$.

**Definition 3.8.** An $A_{k,\alpha}$*-structure* on $\widetilde{S}$ is a maximal atlas on $\widetilde{S}$ such that the transition functions for charts contained in $S \subseteq \widetilde{S}$ are $C^k$, and in local coordinates, in a neighborhood of $i^0 = 0$, the transition functions belong to $A_{k,\alpha}(B_R(0))$ for some $R > 0$.

Clearly, $A_{k+1,\alpha}(B_R(0)) \subseteq A_{k,\alpha}(B_R(0))$ and $C^{k+1}(S) \subseteq C^k(S)$, so $A_{k+1,\alpha}$-transition functions are also $A_{k,\alpha}$-transition functions. In particular, an $A_{k+1,\alpha}$-atlas is an $A_{k,\alpha}$-atlas. If $S$ is smooth, we can define such an $A_{k,\alpha}$-structure on $\widetilde{S}$ for any $k \geq 1$, and we could also define an $A_{\infty,\alpha}$-atlas by demanding it is $A_{k,\alpha}$ for any $k$.

**Definition 3.9.** The $A_{k,\alpha}$*-functions* on $\widetilde{S}$ are functions that are $C^k$ on $S \subseteq \widetilde{S}$ and its coordinate representation around $i^0$ belongs to $A_{k,\alpha}(B_R(0))$ for some $R > 0$. We will denote this space of functions by $A_{k,\alpha}\big(\widetilde{S}\big)$.

**Definition 3.10.** A tensor field $t$ is *of class* $B_{l,\alpha}$ if its components $t_I$ (with $I = i_1 \ldots i_m$) in an $A_{k,\alpha}$-atlas, with $k \geq l + 1$, are $C^l$ on $S$ and there exists a constant $C'$ such that

$$|t_I(y) - t_I(0)| \leq C'r(y)^\alpha, \ldots, |\partial_{i_1} \cdots \partial_{i_\ell} t_I| \leq C'r^{\alpha-\ell},$$

for all $y \in U \setminus \{i^0\}$, where $(U, y)$ is a coordinate chart centered at $i^0$.

With those new regularity classes at hand, we want to redefine our notion of asymptotic flatness of Definition 3.2.

**Definition 3.11.** A three-dimensional Riemannian manifold $(S, h)$ is called *weakly asymptotically flat* if there exists an $A_{k,\alpha}$-manifold $\widetilde{S}$ with $k \geq 3$, endowed with a $B_{k-1,\alpha}$-metric $\widetilde{h}$, and a function $\Omega \in C^1\big(\widetilde{S}\big)$ with $\partial_i\Omega \in A_{1,\alpha}\big(\widetilde{S}\big)$, such that:

(i) $S = \widetilde{S} \setminus \left\{i^0\right\}$ for a single point $i^0 \in \widetilde{S}$, and the inclusion $\iota\colon S \hookrightarrow \widetilde{S}$ is a $C^k$-embedding;
(ii) $\iota^*\widetilde{h} = (\iota^*\Omega)^2 h$;
(iii) $\Omega(i^0) = 0$, $d\Omega_{i^0} = 0$ and $\widetilde{D}(d\Omega)\Big|_{i^0} = 2\widetilde{h}|_{i^0}$.

If we work with $C^k$-structures on $\widetilde{S}$, there can be many inequivalent conformal completions of asymptotically flat spaces. This is an unwanted property because it would be unclear how to relate possible definitions of multipole moments in such inequivalent completions. We want the conformal completion to be a property of the space. For the physics, there is no preference for any $A_{k,\alpha}$-coordinate system, so that this is the appropriate regularity class [28]. For this reason, we redefined our notion of asymptotic flatness and we demand that $\widetilde{S}$ is a $A_{k,\alpha}$-manifold. Luckily, the assumed bounds still allow us to do some calculus on such manifolds. In particular, we can still use results about elliptic partial differential equations. The main reason is that Lipschitz continuous functions are weakly differentiable, which turns out to be sufficient. We can still use the conformal Laplacian.

### Uniqueness of the one-point conformal extension in weaker regularity

The question we will be left with for the remainder of this section is whether, if it exists, the conformal completion $(\widetilde{S}, \widetilde{h})$ is unique. The discussion is based on several lemmas and follows Chruściel [28].

**Lemma 3.12.** *Suppose we have a metric $g$ on $B_{r_0}(0)$ satisfying*

$$|g_{ij}(x) - g_{ij}(0)| \leq Cr^{\alpha}, |\partial_k g_{ij}| \leq Cr^{\alpha-1}, \ldots, |\partial_{i_1} \cdots \partial_{i_\ell} g_{ij}| \leq Cr^{\alpha-\ell}, \tag{3.6}$$

*for some constant $C$, $\ell \geq 1$. Suppose we also have a function $c \in C^k(B_{r_0}(0) \setminus \{0\})$ such that*

$$|c| \leq Cr^{\alpha-2}, |\partial_i c| \leq Cr^{\alpha-3}, \ldots, |\partial_{i_1} \cdots \partial_{i_k} c| \leq Cr^{\alpha-2-k},$$

*$k \geq 0$. Then there exists $0 < r_1 \leq r_0$ and a function $f\colon B_{r_1}(0) \setminus \{0\} \to \mathbb{R}$ that is a weak solution of*

$$(\Delta_g + c)f = 0,$$

*on $B_{r_1}(0) \setminus \{0\}$, and satisfies $\frac{1}{4} \leq f \leq 4$.*

*Sketch of the proof.* We refer to [28, Lemma 2.1] for the full proof. The idea is to use conformal invariance as in Proposition 3.4. The Laplace-Beltrami operator and the function $f$ transform in the same way, and for $c$ we demand the same conformal transformation rule as for $-\frac{1}{8}R$. Introducing $(1 - ar^{\alpha})^2$ as a conformal factor for some constant $a$ will do the job. Then it is possible to find sub- and supersolutions to the elliptic partial differential equation, and a sequence of solutions that must be in between the sub- and supersolution. Using careful estimates and extracting a subsequence, one achieves the result. □

Now, Lemma 3.12 gives a solution on $B_{r_1}(0)\setminus\{0\}$, but we also want to understand its solution at 0, which we identify with $i^0$. This is solved by the lemma below. Its proof is based on the fact that we have bounds on our solution, allowing us to extend it to a weak solution on $B_{r_1}(0)$. We also need some bounds on the derivatives such that $f$ is indeed of the wanted form.

**Lemma 3.13.** *In the setting of Lemma 3.12, $f$ can be extended to a weak solution of*

$$(\Delta_g + c)f = 0,$$

*on $B_{r_1}(0)$. Furthermore, there exist constants $f(0)$ and $C_2$ such that*

$$|f(x) - f(0)| \leq C_2 r^{\alpha}, |\partial_i f| \leq C_2 r^{\alpha-1}, \ldots, |\partial_{i_1} \cdots \partial_{i_m} f| C_2 r^{\alpha-m},$$

*if $0 < \alpha < 1$, and if $\alpha = 1$,*

$$|f(x) - f(0)| \leq C_2 r \log r, |\partial_i f| \leq C_2 \log r, \ldots, |\partial_{i_1} \cdots \partial_{i_m} f| \leq C_2 r^{\alpha-m} \log r,$$

*where $m = \min(\ell, k+1)$.*

*Proof.* See [28, Lemma 2.2]. □

Next, we want to understand the conformal factor a bit better. To this end, we utilise the elliptic partial differential operator (3.2) together with Lemma 3.12 and Lemma 3.13.

**Proposition 3.14.** *Let $0 < \alpha < 1$, and let $g^1$ and $g^2$ be two metric on neighborhoods $U_1 \subseteq \mathbb{R}^3$ and $U_2 \subseteq \mathbb{R}^3$, respectively, of the origin, and suppose we have a homeomorphism $\varphi \colon U_1 \to U_2$ that maps 0 to 0 and is $C^1$ away from the origin, and the metrics are related by*

$$g^1_{ij}(x) = \omega^2(x) g^2_{kl}(\varphi(x)) \frac{\partial \varphi^k}{\partial x^i} \frac{\partial \varphi^l}{\partial x^j}.$$

*Moreover, suppose $g^1$ and $g^2$ obey the inequalities from equation (3.6) with $\ell \geq 2$. Then the function $\omega$ can be extended to a continuous strictly positive function on $U_1$, satisfying*

$$|\partial_i \omega| \leq C r^{\alpha-1},$$

*for some constant $C$.*

*Sketch of the proof.* For more details, we refer to Chruściel [28, Proposition 2.3]. We restrict ourselves to $\ell \geq 3$, and $\ell = 2$ can be done by an approximation argument. By Lemma 3.12 there exist functions $\phi_1$ and $\phi_2$ between $\frac{1}{4}$ and 4 such that

$$\left( \Delta_{g^a} - \frac{1}{8} \widetilde{R}_a \right) \widetilde{\phi}_a = 0,$$

for $a = 1, 2$. Consider the metrics $\widetilde{g}^a = \phi_a^4 g^a$, then the corresponding Ricci scalars vanish, i.e. $\widetilde{R}_a = 0$. Moreover, Lemma 3.13 tells us that the metrics $\widetilde{g}^a$ also satisfy the inequalities from equation (3.6). Let $\Omega = \phi_2^{-2} \omega \phi_1^2$, then $\widetilde{g}^1 = \Omega^2 \widetilde{g}^2$. Hence, we must have $\Delta_{\widetilde{g}^1} \Omega^{\frac{1}{2}} = 0$ and $\Delta_{\widetilde{g}^2} \Omega^{-\frac{1}{2}} = 0$ because $\widetilde{R}_1 = \widetilde{R}_2 = 0$. But then, like in the proof of Lemma 3.5, $\Omega^{\frac{1}{2}}$ must be bounded or go as $\frac{1}{r}$ by Serrin [100], and the same holds for $\Omega^{-\frac{1}{2}}$. We do not want that $\Omega^{-\frac{1}{2}}$ vanishes at the origin, giving that $\Omega^{\frac{1}{2}}$ must be bounded, and replacing the roles shows that $\Omega^{-\frac{1}{2}}$ must be bounded. But then Lemma 3.13 gives estimates on the derivatives of $\Omega^{\frac{1}{2}}$, so we can bound the derivatives of $\Omega^{\frac{1}{2}}$, $\phi_1$ and $\phi_2$, giving the result for $\omega$. □

This proposition will be important to prove uniqueness of the conformal extensions in the definition of weak asymptotic flatness. The following theorem states the result.

**Theorem 3.15.** *Suppose $(\widetilde{S}_1, i^0_1, \widetilde{g}^1)$ and $(\widetilde{S}_2, i^0_2, \widetilde{g}^2)$ are two $A_{k,\alpha}$ manifolds with $B_{k-1,\alpha}$ metrics for some $0 < \alpha < 1$, $k \geq 3$. Suppose there is a pointed continuous conformal mapping between $(\widetilde{S}_1, i^0_1, \widetilde{g}^1)$ and $(\widetilde{S}_2, i^0_2, \widetilde{g}^2)$ that is differentiable on $\widetilde{S}_1 \setminus \{i^0_1\}$, then it is $A_{k,\alpha}$.*

*Sketch of the proof.* We refer to [28, Theorem 2.4] for a full proof. The proof basically works by writing out the conformal transformation laws for the Ricci tensor and the Ricci scalar. Then the equation for the Ricci scalar gives that $\Delta_{g^1} \Omega$ is bounded by $C r^{\alpha-2}$, which implies by the other equation that the second order partial derivatives of $\Omega$ satisfy the same bound, with a possibly different constant $C$. Using the conformal transformation for the Christoffel symbols, we see that the the coordinate transformation is $A_{3,\alpha}$. Taking sufficiently many derivatives of all these equations gives the result. □

Since weak asymptotic flatness does not guarantee that we can differentiate at $i^0$ many times, it is not sufficient for us to define multipole moments. If we allow ourselves to take directional derivatives at $i^0$, the uniqueness is spoiled. Then there are logarithmic ambiguities [28]. However, these logarithmic ambiguities do not affect the four-momentum [3]. The main reason why these $A_{k,\alpha}$-structures are useful is how they correspond to versions of coordinate-wise asymptotic flatness [9, 27, 28, 71, 85].

# Part II

# Multipole Moments in Vacuum

# Chapter 4

# Geroch–Hansen formalism

The goal of this chapter is to construct multipole moments in general relativity in a geometric way. We restrict ourselves to stationary asymptotically flat, vacuum solutions of the Einstein equations. There are several ways to define multipole moments and we study the one by Geroch–Hansen [42, 48]. To define the multipole moments geometrically, we utilise the mass and angular momentum potentials, which we introduce in Section 4.1. After that, we present a rigorous construction for the multipole moments in Section 4.2. However, it is often difficult to calculate the multipole moments, but we can simplify the construction in axisymmetric spacetimes as we will see in Section 4.3. This allows us to compute the multipole moments for the Kerr solution to arbitrary order.

Remember that we work on a stationary spacetime $(M, g)$ with stationary vector field $\xi$ and observer space $S$, which is constructed in Section 2.2. In this chapter, we also assume $M$ is a vacuum solution of the Einstein equations. The observer space is a three-dimensional Riemannian manifold with a metric $h$ determined (2.2) and we assume $(S, h)$ is asymptotically flat, giving the Riemannian manifold $\big(\widetilde{S}, \widetilde{h}\big)$ and scalar field $\Omega \in C^2\big(\widetilde{S}\big)$ as in Definition 3.2. We assume the conditions of Theorem 3.3 hold such that $\big(\widetilde{S}, \widetilde{h}\big)$ is unique. Since we are only interested in the local picture around $i^0$, we can remove the set $K$ in Theorem 3.3 from $S$ and $\widetilde{S}$. We will do this for simplicity. Correspondingly, one could also remove $\pi^{-1}(K)$ from $M$, where $\pi\colon M \to S$ is the projection. So, we assume that $S$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{\mathbb{B}}^3 \cong \mathbb{B}^3 \setminus \{0\}$ and $\widetilde{S} \cong \mathbb{B}^3$. For the de Rham cohomologies of $S$, this gives $H^1_{\mathrm{dR}}(S) = 0$ and $H^2_{\mathrm{dR}}(S) = \mathbb{R}$ because $\mathbb{B}^3 \setminus \{0\}$ is homotopy equivalent to $\mathbb{S}^2$ [72, Chapter 17]. Like in the previous chapter, we drop the primes when working on $S$ and it should be clear where we are working.

## 4.1 Mass and angular momentum potentials

In this section, we introduce the mass and angular momentum potential. The potentials turn out to be functions on $S$ and we bring them to $\widetilde{S}$ with a conformal factor. The main result of this section is proving that the mass and angular momentum potentials are indeed smooth on $\widetilde{S}$ (under an extra assumption).

### Potentials on $S$

Before we define the multipole moments, we recall two important tensor fields from Chapter 2. Firstly, equation (2.1) defines a scalar field $\lambda$ as

$$\lambda = -g(\xi, \xi),$$

which reduces to a scalar field on $S$. The other one is the twist one-form $\omega$ defined in Definition 2.9. In vacuum, the twist one-form is closed by Theorem 2.12. Since we assume $H^1_{\mathrm{dR}}(S) = 0$ (see the paragraph above the header of this section), there exists a twist potential $f \in C^\infty(S)$ such that

$$df = \omega.$$

The twist potential $f$ is not uniquely determined by this equation. We may add a function $\widetilde{f}$ such that $d\widetilde{f} = 0$ to $f$. Since $S$ is assumed to be connected, $d\widetilde{f} = 0$ implies that $\widetilde{f}$ is constant. We want to take this constant to be $-\lim_{x\to i^0} f(x)$, because that would give $\lim_{x\to i^0}\left(f + \widetilde{f}\right)(x) = 0$. However, it is not clear whether $-\lim_{x\to i^0} f(x)$ exists. This is an extra assumption and we come back to it at the end of this section. For now, $f$ is just a primitive of $\omega$.

**Definition 4.1.** The *mass potential* on $S$ is given by[6]

$$\phi_M = \frac{1 - \lambda^2 - f^2}{4\lambda}, \tag{4.1}$$

and the *angular momentum potential* is given by

$$\phi_J = \frac{-f}{2\lambda}. \tag{4.2}$$

They are analogous to the Newtonian mass and angular momentum potentials [48]. Sometimes, it is convenient to view the mass and angular momentum potentials as one complex potential via

$$\phi_C = \phi_M + i\phi_J.$$

Alternatively, we can consider the Ernst potential [36]. Then we write $\mathcal{E} = \lambda + if$ and we consider the potential given by

$$\phi_E = \frac{1+\mathcal{E}}{1-\mathcal{E}} = \frac{4\lambda}{(1-\lambda)^2 + f^2}\phi_C. \tag{4.3}$$

All these potentials are defined by functions on $S$, so we see that the potentials itself are also functions on $S$.

Besides closedness of $\omega$, the Einsteins equations in vacuum also imply that

$$\Delta_h \lambda = \lambda^{-1}|d\lambda|_h^2 - \lambda^{-1}|df|_h^2, \tag{4.4}$$

and that Ricci tensor on $(S, h)$ is

$$Rc = \frac{1}{2}\lambda^{-2}(df \otimes df + d\lambda \otimes d\lambda) = \frac{1}{2}\lambda^{-2}\big(df^2 + d\lambda^2\big). \tag{4.5}$$

[6]We use the opposite sign of Hansen [48] to ensure that the mass monopole moment in the Schwarzschild spacetime returns the mass parameter, not minus the mass.

These equations are found using (2.18) and (2.19). Taking the trace gives the Ricci scalar

$$R = \frac{1}{2}\lambda^{-2}\Big(|df|_h^2 + |d\lambda|_h^2\Big). \tag{4.6}$$

We want to express the Einstein equations in terms of the mass and angular momentum potentials following Hansen [48] and Beig and Simon [13, 103]. In (4.5), we expressed the Ricci tensor of $S$ in terms of $\lambda$ and $f$. We can also express it in terms of $\phi_M$ and $\phi_J$:

**Lemma 4.2.** *The Ricci tensor of $(S, h)$ satisfies*

$$Rc = 2(d\phi_M \otimes d\phi_M + d\phi_J \otimes d\phi_J - d\Sigma \otimes d\Sigma) = 2\big(d\phi_M^2 + d\phi_J^2 - d\Sigma^2\big), \tag{4.7}$$

*where* $\Sigma = \frac{1}{2}\big(1 + 4\phi_M^2 + 4\phi_J^2\big)^{\frac{1}{2}} = \big(\frac{1}{4} + \phi_M^2 + \phi_J^2\big)^{\frac{1}{2}} = \frac{1}{4}\lambda^{-1}\big(\lambda^2 + f^2 + 1\big)$.

*Proof.* We have

$$d\Sigma = \frac{\phi_M d\phi_M + \phi_J d\phi_J}{\Sigma}, \tag{4.8}$$

so

$$\begin{aligned} d\Sigma^2 &= \Sigma^{-2}\big(\phi_M^2 d\phi_M^2 + 2\phi_M\phi_J d\phi_M d\phi_J + \phi_J^2 d\phi_J^2\big) \\ &= \left(1 - \frac{1 + 4\phi_J^2}{1 + 4\phi_M^2 + 4\phi_J^2}\right) d\phi_M^2 + \frac{8\phi_M\phi_J}{1 + 4\phi_M^2 + 4\phi_J^2} d\phi_M d\phi_J + \left(1 - \frac{1 + 4\phi_M^2}{1 + 4\phi_M^2 + 4\phi_J^2}\right) d\phi_J^2. \end{aligned}$$

For the right-hand side of equation (4.7), this gives

$$d\phi_M^2 + d\phi_J^2 - d\Sigma^2 = \frac{1 + 4\phi_J^2}{1 + 4\phi_M^2 + 4\phi_J^2} d\phi_M^2 - \frac{8\phi_M\phi_J}{1 + 4\phi_M^2 + 4\phi_J^2} d\phi_M d\phi_J + \frac{1 + 4\phi_M^2}{1 + 4\phi_M^2 + 4\phi_J^2} d\phi_J^2.$$

Using the expression for $\phi_M$ and $\phi_J$, we have

$$d\phi_M = \frac{1}{4}\lambda^{-2}\big(\lambda^2 - f^2 + 1\big) d\lambda + \frac{1}{2}\lambda^{-1} f df,$$

and

$$d\phi_J = \frac{1}{2}\lambda^{-1} df - \frac{1}{2}\lambda^{-2} f d\lambda.$$

A tedious but straightforward calculation shows that

$$d\phi_M^2 + d\phi_J^2 - d\Sigma^2 = \frac{1}{4}\lambda^{-2}\big(d\lambda^2 + df^2\big).$$

Equation (4.5) finishes the proof. □

In the spirit of (2.16) and (2.17) where we calculated the Laplacian of $\lambda$ and $f$, respectively, we can also calculate the Laplacians of the mass and angular momentum potentials.

**Lemma 4.3.** *The mass and angular momentum potentials satisfy*

$$\left(\Delta_h - \frac{1}{8}R\right)\phi_A = \frac{15}{8}\kappa^4\phi_A, \tag{4.9}$$

*where $\kappa^4 = R$ on $(S, h)$, for $A = M, J$.*

*Proof.* Using equation (2.17), we see that

$$\Delta_h f = 2\lambda^{-1} df(\operatorname{grad}\lambda).$$

Together with equations (4.4) and (4.6), a direct computation shows that

$$\Delta_h \phi_A = 2R\phi_A,$$

for $A = M, J$, giving the result. □

## Potentials on $\widetilde{S}$

The reason to introduce $\kappa$ in Lemma 4.3 is because we want a differential conformal transformation rule for $\kappa = R^{\frac{1}{4}}$ than for the Ricci scalar $R$ itself. Under conformal transformations $\widetilde{h} = \Omega^2 h$, we demand

$$\widetilde{\kappa} = \Omega^{-\frac{1}{2}}\kappa, \qquad \widetilde{\phi}_A = \Omega^{-\frac{1}{2}}\phi_A, \tag{4.10}$$

for $A = M, J$. Equation (4.9) is then conformally invariant by Proposition 3.4:

$$\left(\widetilde{\Delta}_{\widetilde{h}} - \frac{1}{8}\widetilde{R}\right)\widetilde{\phi}_A = \frac{15}{8}\widetilde{\kappa}^4\widetilde{\phi}_A. \tag{4.11}$$

To conclude smoothness of the potentials $\widetilde{\phi}_A$, we want to improve the regularity using the elliptic partial differential equation (4.11). However, the coefficients of this differential equation depend $\widetilde{\kappa}$, which is not necessarily smooth. Therefore, we also want to use an elliptic partial differential equation for $\widetilde{\kappa}$. We introduce the Cotton tensor field $C$, defined by

$$C_{ijk} = D_k R_{ij} - D_j R_{ik} + \frac{1}{4}(h_{ik}D_j R - h_{ij}D_k R).$$

The Cotton tensor field is conformally invariant in dimension 3 [73, Proposition 7.34], reading $\widetilde{C}_{ijk}\widetilde{C}^{ijk} = \Omega^{-6}C_{ijk}C^{ijk}$. Contracting equation (4.7) gives

$$\kappa^4 = R = 2\Big(|d\phi_M|_h^2 + |d\phi_J|_h^2 - |d\Sigma|_h^2\Big). \tag{4.12}$$

Equations (4.9), (4.7) and (4.12) together with a tedious calculation show that $\kappa$ satisfies an elliptic partial differential equation of the form [48, Equation (2.20)]

$$\left(\Delta_h - \frac{1}{8}R\right)\kappa = \frac{3}{8}\kappa^5 + \kappa^{-7}\left(\frac{1}{4}C_{ijk}C^{ijk} + F\big(\phi_A, D\phi_A, D^2\phi_A, \Sigma, D\Sigma, D^2\Sigma, \kappa, D\kappa\big)\right). \tag{4.13}$$

The function $F$ can be shown to be conformally invariant in the sense that $\widetilde{F} = \Omega^{-6}F$ when $\widetilde{\Sigma} = \Sigma$. Hence, Proposition 3.4 implies that (4.13) is also conformally invariant

$$\left(\widetilde{\Delta}_{\widetilde{h}} - \frac{1}{8}\widetilde{R}\right)\widetilde{\kappa} = \frac{3}{8}\widetilde{\kappa}^5 + \widetilde{\kappa}^{-7}\left(\frac{1}{4}\widetilde{C}_{ijk}\widetilde{C}^{ijk} + \widetilde{F}\Big(\widetilde{\phi}_A, \widetilde{D}\widetilde{\phi}_A, \widetilde{D}^2\widetilde{\phi}_A, \widetilde{\Sigma}, \widetilde{D}\widetilde{\Sigma}, \widetilde{D}^2\widetilde{\Sigma}, \widetilde{\kappa}, \widetilde{D}\widetilde{\kappa}\Big)\right). \tag{4.14}$$

However, we also introduced a new function: $\widetilde{\Sigma} = \Sigma$. Using equations (4.8), (4.9), and (4.12), we an easy calculation shows

$$\begin{aligned}\Delta_h\Sigma &= -\Sigma^{-1}|d\Sigma|_h^2 + \Sigma^{-1}\Big(|d\phi_M|_h^2 + \phi_M\Delta_h\phi_M + |d\phi_J|_h^2 + \phi_J\Delta_h\phi_J\Big)\\ &= \frac{1}{2}\Sigma^{-1}\kappa^4 + 2\Sigma^{-1}\kappa^4\big(\phi_M^2 + \phi_J^2\big) = 2\kappa^4\Sigma.\end{aligned}$$

This equation is, however, not conformally invariant. By writing zero in a hard way, this equation can be rewritten as[7]

$$\begin{aligned}\Delta_h \Sigma = 6\kappa^4\Sigma - 2\big(\phi_M^2 + \phi_J^2\big)^{-1} d\Sigma(\phi_M \operatorname{grad}\phi_M + \phi_J \operatorname{grad}\phi_J) \\ + 2\Sigma\big(4\Sigma^2 - 1\big)\big(\phi_M^2 + \phi_J^2\big)^{-2}|\phi_M d\phi_J - \phi_J d\phi_M|_h^2,\end{aligned} \tag{4.15}$$

and from this equation an easy calculation shows

$$\begin{aligned}\widetilde{\Delta}_{\widetilde{h}}\widetilde{\Sigma} = 6\widetilde{\kappa}^4\widetilde{\Sigma} - 2\Big(\widetilde{\phi}_M^2 + \widetilde{\phi}_J^2\Big)^{-1} d\widetilde{\Sigma}\Big(\widetilde{\phi}_M \operatorname{grad}\widetilde{\phi}_M + \widetilde{\phi}_J \operatorname{grad}\widetilde{\phi}_J\Big) \\ + 2\widetilde{\Sigma}\Big(4\widetilde{\Sigma}^2 - 1\Big)\Big(\widetilde{\phi}_M^2 + \widetilde{\phi}_J^2\Big)^{-2}\Big|\widetilde{\phi}_M d\widetilde{\phi}_J - \widetilde{\phi}_J d\widetilde{\phi}_M\Big|_{\widetilde{h}}^2.\end{aligned} \tag{4.16}$$

Equations (4.11), (4.14) and (4.16) constitute elliptic partial differential equations for $\widetilde{\phi}_A$, $\widetilde{\kappa}$ and $\widetilde{\Sigma}$, respectively, whose coefficients depend on each other. We can show that they are smooth by exploiting the bootstrap of elliptic regularity [48]:

**Lemma 4.4.** *Let $(M, g)$ be a stationary asymptotically flat spacetime with Ricci tensor $Rc = 0$. Suppose $\widetilde{\phi}_M$, $\widetilde{\phi}_J$ and $\widetilde{\kappa}$ extend to $C^2$-functions on $\widetilde{S}$ such that $\widetilde{\phi}_M(i^0) \neq 0$, then $\widetilde{\Sigma}$ continuously extends to a function on $\widetilde{S}$ and all four functions are smooth on $\widetilde{S}$.*

*Proof.* For $\widetilde{\Sigma}$, we can write

$$\widetilde{\Sigma} = \left(\frac{1}{4} + \Omega\widetilde{\phi}_M^2 + \Omega\widetilde{\phi}_J^2\right)^{\frac{1}{2}},$$

which is positive, so it is a $C^2$-function on $\widetilde{S}$. Equations (4.11), (4.14) and (4.16) constitute elliptic partial differential equations on $S = \widetilde{S} \setminus \{i^0\}$ with respect to $\widetilde{h}$. By continuity and the $C^2$-assumptions, they extend to equations on $\widetilde{S}$.

We proceed by induction. Suppose $\widetilde{\phi}_M$, $\widetilde{\phi}_J$, $\widetilde{\kappa}$ and $\widetilde{\Sigma}$ are $C^n$-functions for some $n \geq 2$. Then (4.11) and (4.16) constitute elliptic partial differential equations for $\widetilde{\phi}_A$ and $\widetilde{\Sigma}$ whose coefficients are $C^{n-1}$-functions, so $\widetilde{\phi}_A$ and $\widetilde{\Sigma}$ are $C^{n+1}$-functions on $\widetilde{S}$ for $A = M, J$. But then (4.14) constitutes an elliptic partial differential equation for $\widetilde{\kappa}$ whose coefficients are $C^{n-1}$, so $\widetilde{\kappa}$ is also a $C^{n+1}$-function on $\widetilde{S}$. By induction, the functions are smooth on $S$. □

Since we need smoothness of the potentials to define multipole moments up to arbitrary order, we are only interested in the situation where this is satisfied. The previous lemma shows that it is sufficient to require the potentials and $\widetilde{\kappa}$ extend to $C^2$-functions on $\widetilde{S}$. This is the reason why Beig and Simon include in the definition of asymptotic flatness that the potentials must extend to $C^2$-functions on $\widetilde{S}$ [12, 13].

As promised, we have another look at the gauge freedom in the twist potential. We assume that $\widetilde{\phi}_M = \Omega^{-\frac{1}{2}}\phi_M$ and $\widetilde{\phi}_J = \Omega^{-\frac{1}{2}}\phi_J$ extend to $i^0$ in a $C^2$ (or smooth) sense, which implies that

$$\lim_{x \to i^0} \phi_M(x) = 0, \qquad \lim_{x \to i^0} \phi_J = 0.$$

Therefore, $\phi_M$ and $\phi_J$ extend continuously to $i^0$ with 0. Suppose there is a sequence $(x_n)$ in $S$ such that $x_n \to i^0$ and $\lambda(x_n) \to \infty$. Then we have $\phi_M(x_n) \to -\infty$, which contradicts that

[7] This is equation (2.21) in [48], but we corrected a sign error.

$\lim_{x\to i^0}\phi_M(x) = 0$. Therefore, we can safely assume there is a neighborhood $U \subseteq \widetilde{S}$ of $i^0$ such that $\lambda$ is bounded on $U \setminus \{i^0\}$. In that case, $\lim_{x\to i^0}\phi_J = 0$ implies that

$$\lim_{x\to i^0} f = 0.$$

This is precisely how we want to fix the gauge in the discussion above Definition 4.1. If $\widetilde{\phi}_M$ and $\widetilde{\phi}_J$ are smooth, the gauge is fixed! Moreover, the facts that $\lambda$ is positive and $\lim_{x\to i^0}\phi_M = 0$ imply

$$\lim_{x\to i^0} \lambda = 1.$$

Below Definition 4.1, we also saw the potentials $\phi_C$ and $\phi_E$, which differ by the factor $\frac{4\lambda}{(1+\lambda)^2+f^2}$. Observe that

$$\lim_{x\to i^0} \frac{4\lambda}{(1+\lambda)^2+f^2} = 1.$$

When we bring $\phi_C$ and $\phi_E$ to $\widetilde{S}$ using $\widetilde{\phi}_A = \Omega^{-\frac{1}{2}}\phi_A$, we can therefore take this factor $\frac{4\lambda}{(1+\lambda)^2+f^2}$ into the conformal factor and the resulting multipole moments should be equivalent. Note that Theorem 3.3 leaves the freedom to change the conformal factor by a function that is 1 at $i^0$. Proposition 4.10 in the next section discusses how the multipole moments change when we change the conformal factor.

Note that if the mass of the system vanishes, i.e., if $\widetilde{\phi}_M\left(i^0\right) = 0$, then the proof of the above lemma cannot be applied because $\left(\widetilde{\phi}_M^2 + \widetilde{\phi}_J^2\right)^{-1}$ is typically not smooth at $i^0$ anymore. In that case, a more delicate analysis is needed. We do not investigate this issue but it is an interesting open problem to find out whether results like Lemma 4.4 exist when $\widetilde{\phi}_M(i^0)^2 + \widetilde{\phi}_J(i^0)^2 = 0$.[8]

## 4.2 Multipole moments

Finally, we have the tools at hand to define the Geroch–Hansen multipole moments. The multipole moments are inductively defined tensors on $\widetilde{S}$ evaluated at $i^0$. We follow the approach of Hansen [48].

**Definition 4.5.** Let $(S, h)$ be an asymptotically flat Riemannian manifold with $\left(\widetilde{S}, \widetilde{h}\right)$ and $\Omega$ as in Definition 3.2. Let $\phi$ be a smooth function on $S$ such that $\widetilde{\phi} = \Omega^{-\frac{1}{2}}\phi$ extends smoothly to $\widetilde{S} = S \cup \{i^0\}$. The *sequence* $\left(P^k\right)_{k\in\mathbb{N}_0}$ *of symmetric trace-free covariant $k$-tensor fields of* $\phi$ on $\widetilde{S}$ is defined by $P^0 = \widetilde{\phi}$ and

$$P^{k+1} = \left(\widetilde{D}P^k - \frac{1}{2}k(2k-1)P^{k-1} \otimes \widetilde{Rc}\right)^{STF}, \tag{4.17}$$

for $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $T^STF$ denotes taking the totally symmetric and trace-free part of $T$ and $\widetilde{Rc}$ denotes the Ricci tensor on $\left(\widetilde{S}, \widetilde{h}\right)$. The $2^k$*-pole moment of* $\phi$ is $P^k\big|_{i^0}$.

Finding the symmetric trace-free part of a covariant tensor field goes in two steps: first we take the symmetric part and then the trace-free part. We denote the symmetric part of $T$ by

[8] At least, I am not aware of such results.

$T^S$. If $T$ is a covariant $k$-tensor field, this is given by

$$T^S(X_1,\dots,X_k)=\frac{1}{k!}\sum_{\sigma\in S_k}T(X_{\sigma(1)},\dots,X_{\sigma(k)}),$$

or in coordinates

$$T^S_{i_1\dots i_k}=T_{(i_1\dots i_k)}=\frac{1}{k!}\sum_{\sigma\in S_k}T_{i_{\sigma(1)}\cdots i_{\sigma(k)}}. \tag{4.18}$$

For the trace-free part, the explicit formula is a bit uglier. The trace-free part of a symmetric tensor is of the form

$$T^{STF}_{i_1\dots i_k}=\sum_{m=0}^{\lfloor\frac{k}{2}\rfloor}A^k_m h_{(i_1i_2}\cdots h_{i_{2m-1}i_{2m}}T^S_{i_{2m+1}\dots i_k)j_1\dots j_{2m}}h^{j_1j_2}\cdots h^{j_{2m-1}j_{2m}}, \tag{4.19}$$

with $A^0_0=1$ and the other constants $A^k_m$ are characterised by demanding that $h^{i_1i_2}T^{STF}_{i_1\dots i_k}=0$. Then, these constants $A^k_m$ must be given by [18, Appendix A]

$$A^k_m=\frac{(-1)^m k!(2k-2m-1)!!}{2^m m!(k-2m)!(2k-1)!!}.$$

Here, $n!!$ denotes the double factorial. It is recursively defined by $(-1)!!=0!!=1$ and $n!!=n\cdot(n-2)!!$. If $n=2m$, this gives

$$(2m)!!=2m(2m-2)\cdots 2=2^m m!,$$

and if $n=2m-1$, this gives

$$(2m-1)!!=(2m-1)(2m-3)\cdots 1=\frac{(2m)!}{2^m m!}, \tag{4.20}$$

for $m\geq 1$. One of the nice properties of $(-1)!!$ is that the formula for $A^k_m$ also returns $A^0_0=1$.

Before we continue our discussion about multipole moments, we want to observe a few properties about taking the symmetric trace-free part of a tensor. The first one is that taking the symmetric trace-free part is a linear operation, which follows immediately from (4.18) and (4.19). Secondly, we observe that the symmetric trace-free part of tensors of the form $T\otimes h$ vanish.

**Lemma 4.6.** *Let $(S,h)$ be a three-dimensional Riemannian manifold and let $T$ be a covariant $k$-tensor field on $S$, then $(T\otimes h)^{STF}=0$.*

*Proof.* Let $T$ be a covariant $k$-tensor field and write $\overline{T}=T\otimes h$, then the symmetric part is

$$\overline{T}^S_{i_1\dots i_{k+2}}=T_{(i_1\dots i_k}h_{i_{k+1}i_{k+2})}=T^S_{(i_1\dots i_k}h_{i_{k+1}i_{k+2})}.$$

Some combinatorics and the fact that $T^S$ and $h$ are symmetric give

$$
\begin{aligned}
&h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} \overline{T}^S_{i_{2m+1} \dots i_{k+2}) j_1 \dots j_{2m}} h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}} \\
&\quad = \frac{2m}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} T^S_{i_{2m+1} \dots i_{k+2}) j_1 \dots j_{2m-2}} h_{j_{2m-1} j_{2m}} \\
&\qquad\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}} \\
&\qquad + \frac{2m(2m-2)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} T^S_{i_{2m+1} \dots i_{k+2}) j_1 \dots j_{2m-3} j_{2m-1}} h_{j_{2m-2} j_{2m}} \\
&\qquad\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}} \\
&\qquad + \frac{4m(k-2m+2)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} T^S_{i_{2m+1} \dots i_{k+1} | j_1 \dots j_{2m-1}} h_{| i_{k+2}) j_{2m}} \\
&\qquad\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}} \\
&\qquad + \frac{(k-2m+2)(k-2m+1)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} h_{i_{2m+1} i_{2m+2}} T^S_{i_{2m+3} \dots i_{k+2}) j_1 \dots j_{2m}} \\
&\qquad\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}} \\
&\quad = \frac{2m(2k-2m+5)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} T^S_{i_{2m+1} \dots i_{k+2}) j_1 \dots j_{2m-2}} h^{j_1 j_2} \dots h^{j_{2m-3} j_{2m-2}} \\
&\qquad + \frac{(k-2m+2)(k-2m+1)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m+1} i_{2m+2}} T^S_{i_{2m+3} \dots i_{k+2}) j_1 \dots j_{2m}} \\
&\qquad\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}}.
\end{aligned}
$$

So, we have two types of terms. In (4.19) applied to $\overline{T}$, we sum over $m$. Note that the first type of terms vanishes if $m = 0$ and the second type vanishes if $m = \lfloor \frac{k+2}{2} \rfloor = \lfloor \frac{k}{2} \rfloor$. Only taking the first type of terms and shifting the summation gives

$$
\begin{aligned}
&\sum_{m=1}^{\lfloor \frac{k+2}{2} \rfloor} \frac{(-1)^m (k+2)!(2k-2m+3)!!}{2^m m!(k-2m+2)!(2k+3)!!} \frac{2m(2k-2m+5)}{(k+2)(k+1)} h_{(i_1 i_2} \cdots h_{i_{2m-1} i_{2m}} T^S_{i_{2m+1} \dots i_{k+2}) j_1 \dots j_{2m-2}} \\
&\quad \cdot h^{j_1 j_2} \dots h^{j_{2m-3} j_{2m-2}} \\
&\quad = -\sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m k!(2k-2m+3)!!}{2^m m!(k-2m)!(2k+3)!!} h_{(i_1 i_2} \cdots h_{i_{2m+1} i_{2m+2}} T^S_{i_{2m+3} \dots i_{k+2}) j_1 \dots j_{2m}} \\
&\qquad \cdot h^{j_1 j_2} \dots h^{j_{2m-1} j_{2m}},
\end{aligned}
$$

which is precisely what we would get when summing over the second type of terms with a minus sign. Therefore, the summations cancel against each other and $\overline{T}^{STF} = 0$. □

Thirdly, we want to know how the symmetric trace-free part behaves on (some) tensor products.

**Lemma 4.7.** *Let $(S, h)$ be a three-dimensional Riemannian manifold and let $R$ and $T$ be covariant tensor fields on $S$, then*

$$\left(R \otimes T^{STF}\right)^{STF} = \left(R^{STF} \otimes T\right)^{STF} = (R \otimes T)^{STF}. \tag{4.21}$$

*Proof.* We only check that $\left(R^{STF}\otimes T\right)^{STF}=(R\otimes T)^{STF}$ and the other equality follows in the same way. By (4.19), we see that $T^{STF}=T^S+\left(\widehat{T}\otimes h\right)^S$ for some symmetric tensor $\widehat{T}$. It is easy to check that $\left(R\otimes T^S\right)^S=(R\otimes T)^S$, which gives

$$\left(R\otimes T^{STF}\right)^S=\left(R\otimes T^S\right)^S+\left(R\otimes\left(\widehat{T}\otimes h\right)^S\right)^S=(R\otimes T)^S+\left(R\otimes\widehat{T}\otimes h\right)^S.$$

Taking the trace-free part yields, using Lemma 4.6,

$$\left(R\otimes T^{STF}\right)^{STF}=(R\otimes T)^{STF}+\left(R\otimes\widehat{T}\otimes h\right)^{STF}=(R\otimes T)^{STF},$$

which proves the result. □

In particular, this result implies that $\left(T^{STF}\right)^{STF}=T^{STF}$. The last observation is about the covariant derivative on symmetric trace-free parts.

**Lemma 4.8.** *Let $(S,h)$ be a three-dimensional Riemannian manifold with total covariant derivative $D$ and let $T$ be a covariant $k$-tensor field on $S$, then $\left(D\left(T^{STF}\right)\right)^{STF}=(DT)^{STF}$.*

*Proof.* From the definition of the covariant derivative on tensor fields, one can directly prove that

$$D_X\left(T^S\right)=(D_XT)^S.$$

For the total covariant derivative, we would also have to symmetrise on the left-hand side once again, giving

$$\left(D\left(T^S\right)\right)^S=(DT)^S.$$

Breaking off the $m=0$ term from the summation in (4.19) shows that

$$T^{STF}=T^S+\left(h\otimes\widehat{T}\right)^S,$$

for some symmetric tensor $\widehat{T}=\widehat{T}^S$. Therefore, applying $D$ to $T^{STF}$ gives

$$\begin{aligned}\left(D\left(T^{STF}\right)\right)^{STF}&=\left(D\left(T^S\right)\right)^{STF}+\left(\left(h\otimes D\widehat{T}\right)^S\right)^{STF}=\left((DT)^S\right)^{STF}+\left(h\otimes D\widehat{T}\right)^{STF}\\&=(DT)^{STF},\end{aligned}$$

where we used metric-compatibility of the Levi-Civita connection in the first equality and we used Lemma 4.6 in the last equality. □

We continue our discussion on multipole moments. In principle, the recursion (4.17) with $P^0=\widetilde{\phi}$ allows for any smooth function $\widetilde{\phi}$ on $\widetilde{S}$. In that case, we pick $\phi=\Omega^{\frac{1}{2}}\widetilde{\phi}$ and this restricts to a smooth function on $S$ because $\Omega$ is smooth and positive on $S$. However, it does not really make sense to apply Definition 4.5 to such arbitrary functions. We want the multipole moments to contain physical information, which is the case for the potentials of Section 4.1.

**Definition 4.9.** Let $(S, h)$ be an asymptotically flat Riemannian manifold whose one-point extension is $(\widetilde{S}, \widetilde{h})$. Let $\phi_M$ and $\phi_J$ be the mass and angular momentum potential, respectively, and suppose $\widetilde{\phi}_A = \Omega^{-\frac{1}{2}}\phi_A$ extends to a smooth function on $\widetilde{S}$ for $A = M, J$. Then the *mass* $2^k$*-pole moment* is the $2^k$-pole moment of $\phi_M$ and is denoted by $M^k$ and the *angular momentum* $2^k$*-pole moment* is the $2^k$-pole moment of $\phi_J$ and is denoted by $J^k$.

To make sure the mass and angular momentum multipole moments are uniquely defined, we need uniqueness of the one-point completion $(\widetilde{S}, \widetilde{h})$ of $(S, h)$ with conformal factor $\Omega$. We assume that the conditions of Theorem 3.3 are satisfied so that there is still some freedom left in the conformal factor. If $\Omega'$ is another conformal factor satisfying all condition in Definition 3.2, then we have $\Omega' = \alpha\Omega$ for some positive function $\alpha \in C^\infty(\widetilde{S})$ with $\alpha(i^0) = 1$. We want to know how the multipole moments transform under such conformal transformations. A formula is given by Beig [11], but it only works for a very specific conformal factor. We show that the result holds more generally:

**Proposition 4.10.** *Let $(S, h)$ be an asymptotically flat Riemannian manifold with one-point extension $(\widetilde{S}, \widetilde{h}_1)$ and conformal factor $\Omega_1$. Let $\alpha$ be a smooth positive function on $\widetilde{S}$ with $\alpha(i^0) = 1$ and let $\phi$ be a smooth function on $S$ such that $\widetilde{\phi}_1 = \Omega_1^{-\frac{1}{2}}\phi$ extends to a smooth function on $\widetilde{S}$. Let $\widetilde{h}_2 = \alpha^2\widetilde{h}_1$, then $(\widetilde{S}, \widetilde{h}_2)$ is also a one-point completion according to Definition 3.2 with conformal factor $\Omega_2 = \alpha\Omega_1$. Let $(P_1^k)$ and $(P_2^k)$ be the sequence of symmetric trace-free covariant $k$-tensor fields of $\phi$ of Definition 4.5 with respect to $\widetilde{h}_1$ and $\widetilde{h}_2$, respectively. Then*

$$P_2^k = \sum_{m=0}^{k} \binom{k}{m} \frac{(2k-1)!!}{(2m-1)!!} (-2)^{-(k-m)} \alpha^{-\frac{1}{2}-(k-m)} \left(P_1^m \otimes d\alpha^{\otimes(k-m)}\right)^{STF}, \tag{4.22}$$

*where $d\alpha^{\otimes n} = d\alpha \otimes \cdots \otimes d\alpha$, the tensor product of $n$ $d\alpha$'s and the double factorial is defined by* (4.20) *and* $(-1)!! = 1$.

*Remark.* Note that it does not matter whether we take the (symmetric) trace-free part $(\cdot)^{STF}$ with respect to $\widetilde{h}_1$ or $\widetilde{h}_2$. There are equally many $h_{ij}$'s and $h^{ij}$'s in each term in (4.19), so in each term the $\alpha$'s cancel. Therefore, it does not matter whether we use $\widetilde{h}_1$ or $\widetilde{h}_2$ in(4.22) because the metrics are conformally related.

*Proof.* We prove the result by induction. Let $\widetilde{D}_1$ and $\widetilde{D}_2$ denote the Levi-Civita connections and let $\widetilde{Rc}_1$ and $\widetilde{Rc}_2$ denote the Ricci tensors with respect to $\widetilde{h}_1$ and $\widetilde{h}_2$, respectively. Since $\Omega_2 = \alpha\Omega_1$, we take $\widetilde{\phi}_2 = \Omega_2^{-\frac{1}{2}}\phi$ and $\widetilde{\phi}_2 = \alpha^{-\frac{1}{2}}\widetilde{\phi}_1$ also extends to a smooth function on $\widetilde{S}$ because $\alpha(i^0) = 1$. Following Definition 4.5, we have $P_1^0 = \widetilde{\phi}_1$ and $P_2^0 = \widetilde{\phi}_2$, so

$$P_2^0 = \alpha^{-\frac{1}{2}} P_1^0.$$

Moreover,

$$P_2^1 = \widetilde{D}_2 P_2^0 = dP_2^0 = \alpha^{-\frac{1}{2}} dP_1^0 - \frac{1}{2}\alpha^{-\frac{3}{2}} P_1^0 d\alpha = \alpha^{-\frac{1}{2}} P_1^1 - \frac{1}{2}\alpha^{-\frac{3}{2}} P_1^0 d\alpha,$$

proving (4.22) for $k = 0, 1$. Assume (4.22) is satisfied for $k - 1$ and $k$ for some $k \in \mathbb{N}$ and we want to prove that it is also satisfied for $k + 1$. Then we want to calculate $P_2^{k+1}$ using

(4.17), so we need $\widetilde{D}_2 P_2^k$ and $P_2^{k-1} \otimes \widetilde{Rc}_2$. Under conformal transformations, the Levi-Civita connection on covariant $k$-tensor fields transforms as [73, Proposition 7.29]

$$\begin{aligned}\widetilde{D}_2 P_2^k(X_1, \dots, X_{k+1}) = \widetilde{D}_1 P_2^k(X_1, \dots, X_{k+1}) &- k\alpha^{-1} X_{k+1}(\alpha) P_2^k(X_1, \dots, X_k) \\ &- \sum_{i=1}^{k} \alpha^{-1} X_i(\alpha) P_2^k(X_1, \dots, X_{i-1}, X_{k+1}, X_{i+1}, \dots, X_k) \\ &+ \sum_{i=1}^{k} \alpha^{-1} h(X_{k+1}, X_i) P_2^k(X_1, \dots, X_{i-1}, \operatorname{grad}_h \alpha, X_{i+1}, \dots, X_k).\end{aligned}$$

When we take the symmetric trace-free part of $\widetilde{D}_2 P_2^k$, the last summation vanishes by Lemma 4.6. Therefore,

$$\left(\widetilde{D}_2 P_2^k\right)^{STF} = \left(\widetilde{D}_1 P_2^k\right)^{STF} - 2k\alpha^{-1}\left(P_2^k \otimes d\alpha\right)^{STF}.$$

By the induction hypothesis, this gives

$$\begin{aligned}\left(\widetilde{D}_2 P_2^k\right)^{STF} = \sum_{m=0}^{k} \binom{k}{m} \frac{(2k-1)!!}{(2m-1)!!} (-2)^{-(k-m)} \alpha^{-\frac{1}{2}-(k-m)} \\ \cdot \Bigg( \left(\widetilde{D}_1 P_1^m \otimes d\alpha^{\otimes(k-m)}\right)^{STF} \\ +(k-m)\left(P_1^m \otimes \widetilde{D}_1(d\alpha) \otimes d\alpha^{\otimes(k-1-m)}\right)^{STF} \\ -\frac{1}{2}(6k-2m+1)\alpha^{-1}\left(P_1^m \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF}\Bigg),\end{aligned} \tag{4.23}$$

where we utilised Lemma 4.7 and Lemma 4.8. The Ricci tensor transforms as [73, Theorem 7.30]

$$\widetilde{Rc}_2 = \widetilde{Rc}_1 - \alpha^{-1}\widetilde{D}_1(d\alpha) - \alpha^{-1}\left(\widetilde{\Delta}_{\widetilde{h}_1}\alpha\right)\widetilde{h}_1 + 2\alpha^{-2} d\alpha \otimes d\alpha,$$

and taking the symmetric trace-free part gives

$$\left(\widetilde{Rc}_2\right)^{STF} = \left(\widetilde{Rc}_1\right)^{STF} - \alpha^{-1}\left(\widetilde{D}_1(d\alpha)\right)^{STF} + 2\alpha^{-2}(d\alpha \otimes d\alpha)^{STF},$$

by Lemma 4.6. By the induction hypothesis for $k-1$,

$$\begin{aligned}\left(P_2^{k-1} \otimes \widetilde{Rc}_2\right)^{STF} = \sum_{m=0}^{k-1} \binom{k-1}{m} \frac{(2k-3)!!}{(2m-1)!!} (-2)^{-(k-1-m)} \alpha^{-\frac{1}{2}-(k-1-m)} \\ \cdot \Bigg( \left(P_1^m \otimes \widetilde{Rc}_1 \otimes d\alpha^{\otimes(k-1-m)}\right)^{STF} \\ -\alpha^{-1}\left(P_1^m \otimes \widetilde{D}_1(d\alpha) \otimes d\alpha^{\otimes(k-1-m)}\right)^{STF} \\ +2\alpha^{-2}\left(P_1^m \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF}\Bigg),\end{aligned} \tag{4.24}$$

where we used Lemma 4.7. For $P_2^{k+1}$, following (4.17), we have

$$P_2^{k+1} = \left(\widetilde{D}_2 P_2^k - \frac{1}{2}k(2k-1)P_2^{k-1} \otimes \widetilde{Rc}_2\right)^{STF} = A + B + C,$$

where, from (4.23) and (4.24), $A$ contains the terms with $\widetilde{D}_1 P_1^m$ and $P_1^m \otimes \widetilde{Rc}_1$, $B$ contains the terms with $\widetilde{D}_1(d\alpha)$, and $C$ contains the other terms which are of the form $P_1^m \otimes d\alpha^{\otimes(k+1-m)}$. Rewriting a little bit easily shows that $B = 0$ because for each $m = 0, \dots, k-1$ the coefficients cancel

$$\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}(k-m) - \frac{1}{2}k(2k-1)\binom{k-1}{m}\frac{(2k-3)!!}{(2m-1)!!}(-2)^{-(k-m-1)}(-1) = 0,$$

as is easily checked by hand or by Mathematica. For $A$, shifting the summation for the $\widetilde{Rc}_1$-terms gives

$$\begin{aligned}
A = &\sum_{m=0}^{k}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}\alpha^{-\frac{1}{2}-(k-m)}\left(\widetilde{D}_1 P_1^m \otimes d\alpha^{\otimes(k-m)}\right)^{STF} \\
&- \frac{1}{2}k\sum_{m=0}^{k-1}\binom{k-1}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-1-m)}\alpha^{-\frac{1}{2}-(k-1-m)}\left(P_1^m \otimes \widetilde{Rc}_1 \otimes d\alpha^{\otimes(k-1-m)}\right)^{STF} \\
= &\sum_{m=0}^{k}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}\alpha^{-\frac{1}{2}-(k-m)}\left(\widetilde{D}_1 P_1^m \otimes d\alpha^{\otimes(k-m)}\right)^{STF} \\
&- \frac{1}{2}k\sum_{m=1}^{k}\binom{k-1}{m-1}\frac{(2k-1)!!}{(2m-3)!!}(-2)^{-(k-m)}\alpha^{-\frac{1}{2}-(k-m)}\left(P_1^{m-1} \otimes \widetilde{Rc}_1 \otimes d\alpha^{\otimes(k-m)}\right)^{STF}.
\end{aligned}$$

Then, we take the summations together and exploiting Lemma 4.7 gives

$$\begin{aligned}
A = &\sum_{m=0}^{l}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}\alpha^{-\frac{1}{2}-(k-m)} \\
&\cdot\left(\left(\widetilde{D}_1 P_1^m - \frac{1}{2}m(2m-1)P_1^{m-1} \otimes \widetilde{Rc}_1\right)^{STF} \otimes d\alpha^{\otimes(k-m)}\right)^{STF}
\end{aligned}$$

Using (4.17) for $m$ and shifting the summation again, we find

$$\begin{aligned}
A &= \sum_{m=0}^{k}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}\alpha^{-\frac{1}{2}-(k-m)}\left(P_1^{m+1} \otimes d\alpha^{\otimes(k-m)}\right)^{STF} \\
&= \sum_{m=1}^{k+1}\binom{k}{m-1}\frac{(2k-1)!!}{(2m-3)!!}(-2)^{-(k+1-m)}\alpha^{-\frac{1}{2}-(k+1-m)}\left(P_1^{m} \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF}.
\end{aligned}$$

Finally, for $C$ we have

$$C = \sum_{m=0}^{k}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k+1-m)}(2(k+m)+1)\alpha^{-\frac{1}{2}-(k+1-m)}\left(P_1^m \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF}.$$

We want to add $A$ and $C$, giving three type of terms: the $m = 0$ term in $C$, the $m = k + 1$ term in $A$ and the terms for $m = 1, \ldots, k$. For the latter, an easy calculation (by hand or by Mathematica) yields

$$\binom{k}{m-1}(2m-1) + \binom{k}{m}(2(k+m)+1) = \binom{k+1}{m}(2k+1).$$

Therefore, we have

$$\begin{aligned}
P_2^{l+1} &= A + C \\
&= (2k+1)!!(-2)^{-(k+1)}\alpha^{-\frac{1}{2}-(k+1)}\left(P_1^0 \otimes d\alpha^{\otimes(k+1)}\right)^{STF} + \alpha^{-\frac{1}{2}}\left(P_1^{k+1}\right)^{STF} \\
&\quad + \sum_{m=1}^{k}\binom{k+1}{m}\frac{(2k+1)!!}{(2m-1)!!}(-2)^{-(k+1-m)}\alpha^{-\frac{1}{2}-(k+1-m)}\left(P_1^m \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF} \\
&= \sum_{m=0}^{k+1}\binom{k+1}{m}\frac{(2k+1)!!}{(2m-1)!!}(-2)^{-(k+1-m)}\alpha^{-\frac{1}{2}-(k+1-m)}\left(P_1^m \otimes d\alpha^{\otimes(k+1-m)}\right)^{STF},,
\end{aligned}$$

proving (4.22) by induction. □

**Corollary 4.11.** *In the setting of Proposition 4.10, the multipole moments transform as*

$$P_2^k\Big|_{i^0} = \sum_{m=0}^{k}\binom{k}{m}\frac{(2k-1)!!}{(2m-1)!!}(-2)^{-(k-m)}\left(P_1^m|_{i^0} \otimes d\alpha|_{i^0}^{\otimes(k-m)}\right)^{STF}. \tag{4.25}$$

*Proof.* The result follows readily from evaluating equation (4.22) at $i^0$ as $\alpha(i^0) = 1$. □

In Newtonian gravity, the multipole expansion depends on the choice of the origin. In the relativistic setting, we can choose the conformal factor. Both transformations should represent the same behaviour. Keeping $\alpha$ close to 1 around $i^0$, we see that the first-order correction of the $2^k$-multipole moment is proportional to $\mathcal{C}\left(P_1^{k-1}\Big|_{i^0} \otimes d\alpha|_{i^0}\right)$. So, the first-order correction only depends on the $2^{k-1}$-multipole moment. This is the same as for Newtonian multipole moments [41, 42]. At first, the $\widetilde{Rc}$-term in (4.17) might seem surprising as the coefficient in an expansion are usually found by taking derivatives. However, this term precisely cancels the correction term that is proportional to $\mathcal{C}\left(P_1^{k-2}\Big|_{i^0} \otimes \widetilde{D}_1(d\alpha)\Big|_{i^0}\right)$. So, if we want a similar behaviour under conformal transformations as for changing the origin for Newtonian multipole moments, we need the $\widetilde{Rc}$-term.

In Newtonian gravity, we often pick the origin such that it lies at the center of mass. That means, if the mass $2^k$-pole moments is the first one that is nonvanishing, then the mass $2^{k+1}$-pole moment does vanish. Assume that the mass of the system is nonvanishing, then the mass monopole moment $M^0$ is nonzero. Then we want the mass dipole moment to vanish. Corollary 4.11 yields

$$M_2^1 = M_1^1 - \frac{1}{2}M_0^1\, d\alpha|_{i^0},$$

so we want to take $\alpha$ such that

$$d\alpha|_{i^0} = \frac{2}{M_0^1}M_1^1.$$

This is always possible and there is still some freedom left in $\alpha$. However, there is no freedom left in the multipole moments anymore. We see that this completely fixes the multipole moments $M_2^k$ and $J_2^k$ by Corollary 4.11.

## 4.3 Axisymmetric spacetimes and the Kerr solution

It is typically very difficult to calculate multipole moments using the Geroch–Hansen formalism. It is still easy to do the calculation for the Schwarzschild spacetime, but the Kerr spacetime is already a hard job. The goal of this chapter is to calculate the multipole moments for the Kerr spacetime to arbitrary order. The main difficulty is that the recursion step becomes computationally heavier each step. The tensors get more components and the expressions get uglier. Luckily, we can simplify the calculations a bit when there are more symmetries.

We start this section with naively starting to calculate the multipole moments for the Kerr spacetime. After it turns out that it is too difficult, we study how the multipole moments simplify when the spacetime is axisymmetric [48, Section 3]. In 1989, Fodor, Hoenselaers and Perjés found a simpler algorithm to calculate multipole moments in axisymmetric spacetimes [37], which we discuss afterwards. At the end of this section, we study an algorithm by Bäckdahl and Herberthson in 2005 [7] that simplifies the job even more. With the last algorithm, we are finally able to compute all multipole moments for the Kerr spacetime.

### Naively computing the multipole moments for the Kerr spacetime

We start with naively doing the calculation, for which we follow Hansen [48, Section 3]. In Boyer–Lindquist coordinates, the Kerr metric looks like

$$\begin{aligned} g = &-\left(1 - \frac{2mr}{r^2 + a^2\cos^2\theta}\right)dt^2 - \frac{4mar\sin^2\theta}{r^2 + a^2\cos^2\theta}dtd\varphi + \frac{r^2 + a^2\cos^2\theta}{r^2 - 2mr + a^2}dr^2 \\ &+ (r^2 + a^2\cos^2\theta)d\theta^2 + \frac{(a^2 + r^2)^2 - a^2(r^2 - 2mr + a^2)\sin^2\theta}{r^2 + a^2\cos^2\theta}\sin^2\theta d\varphi^2, \end{aligned} \tag{4.26}$$

where $m > 0$ and $a \in \mathbb{R}$. Note that we are only interested in what happens for $r > 2m$. We want to interpret $m$ as the mass and $a$ as the scaled angular momentum. The timelike Killing vector field is $\xi = \frac{\partial}{\partial t}$ for $r > 2m$, giving

$$\lambda = -g_{tt} = 1 - \frac{2mr}{r^2 + a^2\cos^2\theta} = \frac{r^2 - 2mr + a^2\cos^2\theta}{r^2 + a^2\cos^2\theta},$$

and $\xi^\flat = g_{t\mu}dx^\mu$. Therefore, the metric (2.2) on the observer space is

$$\begin{aligned} h &= \lambda g + \xi^\flat \otimes \xi^\flat = (-g_{tt}g_{\mu\nu} + g_{t\mu}g_{t\nu})dx^\mu \otimes dx^\nu = (-g_{tt}g_{ij} + g_{ti}g_{tj})dx^i \otimes dx^j \\ &= \frac{r^2 - 2mr + a^2\cos^2\theta}{r^2 - 2mr + a^2}dr^2 + (r^2 - 2mr + a^2\cos^2\theta)d\theta^2 + (r^2 - 2mr + a^2)\sin^2\theta d\varphi^2. \end{aligned}$$

The Levi-Civita tensor is given by

$$\varepsilon = \sqrt{-\det g}dt \wedge dr \wedge d\theta \wedge d\varphi = (r^2 + a^2\cos^2\theta)\sin\theta dt \wedge dr \wedge d\theta \wedge d\varphi.$$

By a tedious calculation involving Christoffel symbols, one finds, either by hand or by Mathematica, that the twist one-form is

$$\omega = \frac{4mar\cos\theta}{(r^2+a^2\cos^2\theta)^2}dr + \frac{2ma(r^2-a^2\cos^2\theta)\sin\theta}{(r^2+a^2\cos^2\theta)^2}d\theta, \tag{4.27}$$

and $\omega = df$ with

$$f = -\frac{2ma\cos\theta}{r^2+a^2\cos^2\theta}.$$

Therefore, the gravitational field potentials are given by

$$\phi_M = \frac{1-\lambda^2-f^2}{4\lambda} = \frac{m(r-m)}{r^2-2mr+a^2\cos^2\theta},$$

and

$$\phi_J = \frac{-f}{2\lambda} = \frac{ma\cos\theta}{r^2-2mr+a^2\cos^2\theta}.$$

Now, we introduce a new coordinate $\overline{R}$ defined by

$$r = \overline{R}^{-1}\Big(1 + m\overline{R} + \frac{1}{4}(m^2-a^2)\overline{R}^2\Big),$$

and the coordinate transformation can be inverted by

$$\overline{R} = \frac{2\Big(r-m-\sqrt{r^2-2mr+a^2}\Big)}{m^2-a^2}.$$

Observe that $\overline{R}$ is positive for $r > 2m$ and $\overline{R} \to 0$ as $r \to \infty$. In the extremal case where $m = |a|$, we have $\overline{R} = \frac{1}{r-m}$ and we also see that $\overline{R}$ is positive and goes to 0 as $r$ goes to infinity. The region $r > 2m$ corresponds to $\overline{R} < \frac{1}{m+|a|}$, so we take $0 < \overline{R} < \frac{1}{m+|a|}$. With this new coordinate we have

$$h = \frac{\Big(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\Big)^2 - a^2\overline{R}^2\sin^2\theta}{\overline{R}^4}d\overline{R}^2 + \frac{\Big(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\Big)^2 - a^2\overline{R}^2\sin^2\theta}{\overline{R}^2}d\theta^2 + \frac{(1-\frac{1}{4}(m^2-a^2)\overline{R}^2)^2}{\overline{R}^2}\sin^2\theta d\varphi^2.$$

Define the function

$$\Omega(\overline{R},\theta,\varphi) = \frac{\overline{R}^2}{\sqrt{\Big(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\Big)^2 - a^2\overline{R}^2\sin^2\theta}}, \tag{4.28}$$

for $0 < \overline{R} < \frac{1}{m+|a|}$ and on this region we have

$$\widetilde{h} = \Omega^2 h = d\overline{R}^2 + \overline{R}^2 d\theta^2 + \frac{\overline{R}^2}{1-\frac{a^2\overline{R}^2\sin^2\theta}{(1-\frac{1}{4}(m^2-a^2)\overline{R}^2)^2}}\sin^2\theta d\varphi^2. \tag{4.29}$$

Now, we want to add a point at infinity, representing $r$ at $\infty$. We already saw that this corresponds to $\overline{R}=0$. So our manifold in $\left(\overline{R},\theta,\varphi\right)$ coordinates is $S=B_{\frac{1}{m+|a|}}(0)\setminus\{0\}\cong\left(0,\frac{1}{m+|a|}\right)\times\mathbb{S}^2$, and our candidate for $\widetilde{S}$ is the open ball $B_{\frac{1}{m+|a|}}(0)$ of radius $\frac{1}{m+|a|}$ centered at the origin in $\mathbb{R}^3$. Since spherical coordinates do not behave very nicely at the origin, it is sometimes useful to give the expression in Cartesian coordinates as well. Let $(x,y,z)$ be the Cartesian coordinates corresponding to $\left(\overline{R},\theta,\varphi\right)$, then we have

$$\Omega=\frac{x^2+y^2+z^2}{\sqrt{(1-\frac{1}{4}(m^2-a^2)(x^2+y^2+z^2))^2-a^2(x^2+y^2)}},$$

and

$$\begin{aligned}\widetilde{h}=\Omega^2h=&\left(1+\frac{a^2y^2}{(1-\frac{1}{4}(m^2-a^2)(x^2+y^2+z^2))^2-a^2(x^2+y^2)}\right)dx^2\\&-\frac{2a^2xy}{(1-\frac{1}{4}(m^2-a^2)(x^2+y^2+z^2))^2-a^2(x^2+y^2)}dxdy\\&+\left(1+\frac{a^2x^2}{(1-\frac{1}{4}(m^2-a^2)(x^2+y^2+z^2))^2-a^2(x^2+y^2)}\right)dy^2+dz^2\\=&\,dx^2+dy^2+dz^2+\frac{a^2}{(1-\frac{1}{4}(m^2-a^2)(x^2+y^2+z^2))^2-a^2(x^2+y^2)}(ydx-xdy)^2.\end{aligned}$$

First, we note that the denominator of $\Omega$ is nonzero in the region where $\overline{R}<\frac{1}{m+|a|}$. We also observe that both expressions can smoothly be extended to the origin, where $\Omega$ vanishes and $\widetilde{h}$ becomes the Euclidean metric. Moreover, $\widetilde{h}$ is positive-definite on the open ball, so $(\widetilde{S},\widetilde{h})$ is indeed a Riemannian manifold. We also have

$$\widetilde{D}\Omega|_{(0,0,0)}=0,$$

and

$$\widetilde{D}\widetilde{D}\Omega|_{(0,0,0)}=2\delta_{ij}dx^i|_{(0,0,0)}\otimes dx^j|_{(0,0,0)}=2\widetilde{h}|_{(0,0,0)}.$$

Therefore, $(\widetilde{S},\widetilde{h})$ satisfies the conditions for asymptotic flatness. In our new coordinates, the gravitational potentials are

$$\phi_M=\frac{m\overline{R}(1+\frac{1}{4}(m^2-a^2)\overline{R}^2)}{\left(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\right)^2-a^2\overline{R}^2\sin^2\theta},$$

and

$$\phi_J=\frac{ma\overline{R}^2\cos\theta}{\left(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\right)^2-a^2\overline{R}^2\sin^2\theta}.$$

Hence, the potentials on $\widetilde{S}$ defined by $\widetilde{\phi}_A=\Omega^{-\frac{1}{2}}\phi_A$ for $A=M,J$ become

$$\widetilde{\phi}_M=\frac{m(1+\frac{1}{4}(m^2-a^2)\overline{R}^2)}{\left(\left(1-\frac{1}{4}(m^2-a^2)\overline{R}^2\right)^2-a^2\overline{R}^2\sin^2\theta\right)^{\frac{3}{4}}},\tag{4.30}$$

and

$$\widetilde{\phi}_J = \frac{ma\overline{R}\cos\theta}{\left(\left(1 - \frac{1}{4}(m^2 - a^2)\overline{R}^2\right)^2 - a^2\overline{R}^2\sin^2\theta\right)^{\frac{3}{4}}}. \tag{4.31}$$

Evaluating them at $\overline{R} = 0$, shows that $M^0 = m$ and $J^0 = 0$. This shows that $m$ is indeed understood as the mass and there is no current monopole. We write the functions in Cartesian coordinates. Taking the derivatives and evaluating at $(0, 0, 0)$ shows that $M^1 = 0$ and $J^1 = madz$. Therefore, $-ma$ can be seen as the angular momentum pointing in the $z$-direction and $M^1 = 0$ tells us that the mass is centered. It is doable to calculate the quadrupole and maybe even octopole moments, but it becomes troublesome at higher orders because of the complicated tensor expressions around $(0, 0, 0)$. Therefore, we need a smarter way to calculate multipole moments for the Kerr space-time.

### Geroch–Hansen formalism in axisymmetric spacetimes

We call a spacetime axisymmetric if there is a spacelike Killing vector field whose integral curves are closed. The closed integral curves are the orbits when "rotating" along the vector field. In stationary spacetimes, we want axisymmetry to nicely work together with stationarity, leading to the following definition:

**Definition 4.12.** A stationary spacetime $(M, g)$ with stationary vector field $\xi$ is called *axisymmetric* if there exists a spacelike Killing vector field $\psi$ whose integral curves are closed and such that $[\xi, \psi] = 0$.

The fact that the Killing vector fields commute, implies that their corresponding flows/isometries also commute. For the Kerr spacetime in Boyer–Lindquist coordinates, the stationary vector field is $\frac{\partial}{\partial t}$ and the axisymmetric vector field is $\frac{\partial}{\partial \varphi}$, so they of course commute. The Kerr spacetime is an example of an axisymmetric stationary spacetime.

Define another vector field $\eta$ by

$$\eta = \psi + \frac{g(\psi, \xi)}{\lambda}\xi = \psi - \frac{g(\psi, \xi)}{g(\xi, \xi)}\xi.$$

Then $\eta$ is a vector field on $M$ such that $\mathcal{L}_\xi \eta = 0$ because $\mathcal{L}_\xi \psi = \mathcal{L}_\xi \xi = 0$ and $\mathcal{L}_\xi g = 0$. Moreover,

$$g(\eta, \xi) = g(\psi, \xi) - \frac{g(\psi, \xi)}{g(\xi, \xi)} g(\xi, \xi) = 0,$$

so $\eta$ is a vector field that lives on the observer space $S$. Furthermore, $\mathcal{L}_\psi \lambda = 0$ because $\mathcal{L}_\psi g = 0$ and $\mathcal{L}_\psi \xi = 0$, which implies that $\mathcal{L}_\eta \lambda = 0$. Moreover,

$$\mathcal{L}_\eta g = d\big(\lambda^{-1} g(\psi, \xi)\big) \otimes \xi^\flat + \xi^\flat \otimes d\big(\lambda^{-1} g(\psi, \xi)\big),$$

because $\mathcal{L}_\xi g = \mathcal{L}_\eta g = 0$ and

$$\mathcal{L}_\eta \xi^\flat = -\lambda d\big(\lambda^{-1} g(\psi, \xi)\big),$$

because $\mathcal{L}_\psi \xi = \mathcal{L}_\xi \xi = 0$. Let $h$ denote our standard Riemannian metric on $S$ given by (2.2), then

$$\mathcal{L}_\eta h = \lambda \mathcal{L}_\eta g + \mathcal{L}_\eta \xi^\flat \otimes \xi^\flat + \xi^\flat \otimes \mathcal{L}_\eta \xi^\flat = 0. \tag{4.32}$$

Therefore, $\eta$ is a Killing vector field on $(S, h)$. We easily see that $\eta$ is $\pi$-related to $\psi$. Since $\pi$ is surjective, we see that the maximal integral curves of $\eta$ in $S$ are precisely the images of the maximal integral curves of $\psi$ in $M$. Since the latter are closed, the former must also be closed. Therefore, $(S, h)$ is axisymmetric with axisymmetric vector field $\eta$.

For the twist one-form $\omega$ given by (2.5) with twist potential $f$ such that $\omega = df$, Cartan's magic formula gives

$$\mathcal{L}_\psi f = i_\psi df = i_\psi \omega. \tag{4.33}$$

In an asymptotically flat vacuum (or, for example, electrovacuum) solution of the Einstein equations that is axisymmetric and stationary, the 2-dimensional planes orthogonal to $\xi$ and $\psi$ are integrable [112, Section 7.1]. In that case, we have coordinates $(t, \varphi, x^2, x^3)$ such that $\xi = \frac{\partial}{\partial t}$, $\psi = \frac{\partial}{\partial \varphi}$ and the metric is of the form

$$(g_{\mu\nu}) = \begin{pmatrix} -\lambda & \lambda w & 0 & 0 \\ \lambda w & g_{\varphi\varphi} & 0 & 0 \\ 0 & 0 & g_{22} & g_{23} \\ 0 & 0 & g_{32} & g_{33} \end{pmatrix}, \tag{4.34}$$

where the functions $\lambda$, $w$, $g_{\varphi\varphi}$, $g_{22}$, $g_{23} = g_{32}$ and $g_{33}$ can only depend on $x^2$ and $x^3$. Therefore, we see that

$$\nabla^\mu \xi^\nu = g^{\mu\rho}\Gamma^\nu_{\rho t} = \frac{1}{2} g^{\mu\rho} g^{\nu\sigma}(\partial_\rho g_{\sigma t} - \partial_\sigma g_{\rho t})$$

vanishes when $(\mu, \nu) = (2, 3)$ or $(\mu, \nu) = (3, 2)$. Therefore,

$$i_\psi \omega = \varepsilon_{\mu\nu\rho\sigma} \psi^\mu \xi^\nu \nabla^\rho \xi^\sigma = \varepsilon_{\varphi t \mu\nu} \nabla^\mu \xi^\nu = 0. \tag{4.35}$$

By (4.33), this implies that $f$ is constant along the integral curves of $\psi$. Together with $\mathcal{L}_\psi \lambda$, this implies that

$$\mathcal{L}_\psi \phi_A = 0,$$

for $A = M, J$, where $\phi_M$ and $\phi_J$ are given by Definition 4.1. Therefore, we also have

$$\mathcal{L}_\eta \phi_A = 0, \tag{4.36}$$

for $A = M, J$.

So, $\eta$ is a vector field on $S$ satisfying (4.32) and (4.36). According to Hansen [48], the conformal factor $\Omega$ can in that case be chosen such that there is a vector field $\widetilde{\eta}$ on $\widetilde{S}$ that equals $\eta$ on $S$, satisfies

$$\mathcal{L}_{\widetilde{\eta}} \widetilde{h} = 0,$$

and so that the axis vector field

$$\widetilde{z}^i = 2\widetilde{\varepsilon}^{ijk} \widetilde{D}_j \widetilde{\eta}_k,$$

is a unit vector field at $i^0$, meaning $\widetilde{h}(\widetilde{z}, \widetilde{z})\Big|_{i^0} = 1$. Here, $\widetilde{\varepsilon}$ is the Levi-Civita tensor for $\left(\widetilde{S}, \widetilde{h}\right)$ and $\widetilde{D}$ is the Levi-Civita connection on $\left(\widetilde{S}, \widetilde{h}\right)$. From (4.32) and (4.36), we can conclude that the gravitational potentials, and hence also the multipole moments, are invariant under the flow of $\widetilde{\eta}$. However, the only direction which is left invariant under this action is spanned by

$\widetilde{z}$. Therefore, the multipole moments must be multiples of $\left(\widetilde{z}^\flat \otimes \cdots \otimes \widetilde{z}^\flat\right)^{STF}|_{i^0}$. We define the constants $C^k$ by

$$P^k|_{i^0} = (2k-1)!!C^k \left(\widetilde{z}^\flat \otimes \cdots \otimes \widetilde{z}^\flat\right)^{STF}\Big|_{i^0}. \tag{4.37}$$

Then we see that

$$C^k = \frac{1}{k!}P^k(\widetilde{z}, \ldots, \widetilde{z})|_{i^0}, \tag{4.38}$$

because $\widetilde{z}^\flat(\widetilde{z}) = 1$ gives

$$\left(\widetilde{z}^\flat \otimes \cdots \otimes \widetilde{z}^\flat\right)^{STF}(\widetilde{z}, \ldots, \widetilde{z})|_{i^0} = \sum_{m=0}^{\lfloor \frac{k}{2} \rfloor} \frac{(-1)^m k!(2k-2m-1)!!}{2^m m!(k-2m)!(2k-1)!!} = \frac{k!}{(2k-1)!!}.$$

Here, the last equality follows by a tedious calculation [37, 107]. So, the information of the multipole moments at $i^0$ is captured in those constants $C^k$.

In case of the mass and angular momentum potentials, we write $C^k = m_k$ when using the mass potential and $C^k = j_k$ when using the angular momentum potential. For the Kerr spacetime, the calculations below (4.31) show that we must have $m_0 = m$, $j_0 = 0$, $m_1 = 0$ and $j_1 = ma$. However, we still do not know the values for higher orders (although the quadrupole and octopole moments are also doable to calculate using the naive method).

### First algorithm to find multipole moments

Now, we know that the multipole moments in axisymmetric stationary vacuum solutions of the Einstein equations can be represented by scalars, but it is still difficult to calculate them. In this part, we discuss the algorithm by Fodor, Hoenselaers and Perjés [37] to do the calculation.

In vacuum, we can simplify the metric in (4.34) to [112, Section 7.1]

$$g = -\lambda(dt - wd\varphi)^2 + \lambda^{-1}(\rho^2 d\varphi^2 + e^{2\gamma}(d\rho^2 + dz^2)), \tag{4.39}$$

where $\xi = \frac{\partial}{\partial t}$ and $\psi = \frac{\partial}{\partial \varphi}$ are the stationary and axisymmetric Killing vector field, respectively, and $\lambda$, $w$ and $\gamma$ are functions that only depend on $\rho$ and $z$. These coordinates are called the Weyl-Lewis-Papapetrou coordinates [87].

Then the metric $h$ on the observer space $S$ looks like

$$h = e^{2\gamma}(d\rho^2 + dz^2) + \rho^2 d\varphi^2.$$

For the one-point completion, we introduce variables $\widetilde{\rho} = \frac{\rho}{\rho^2+z^2}$ and $\widetilde{z} = \frac{z}{\rho^2+z^2}$. Then the metric becomes

$$h = \frac{1}{(\widetilde{\rho}^2 + \widetilde{z}^2)^2}\left(\widetilde{\rho}^2 d\varphi^2 + e^{2\gamma}\left(d\widetilde{\rho}^2 + d\widetilde{z}^2\right)\right).$$

If we take $\Omega = \widetilde{\rho}^2 + \widetilde{z}^2$, we have

$$\Omega^2 h = e^{2\gamma} d\widetilde{\rho}^2 + e^{2\gamma} d\widetilde{z}^2 + \widetilde{\rho}^2 d\varphi^2.$$

We interpret the coordinates $(\widetilde{\rho}, \varphi, \widetilde{z})$ as cylindrical coordinates. Then we want to add the point with $\widetilde{\rho} = 0$ and $\widetilde{z} = 0$ as $i^0$. We need that $\gamma$ vanishes in the limit where $\widetilde{\rho}$ and $\widetilde{z}$ tend to 0. If $\gamma$ does not converge to 0, then the metric is typically not asymptotically flat [37].

Recall the Ernst potential from (4.3)

$$\phi_E = \frac{1-\lambda-if}{1+\lambda+if} = \frac{(1-if)^2-\lambda^2}{(1+\lambda)^2+f^2} = \frac{4\lambda}{(1+\lambda)^2+f^2}(\phi_M + i\phi_J),$$

This potential satisfies the Ernst equation [36, Eq. (11)]

$$\big(\phi_E\overline{\phi_E}-1\big)\Delta\phi_E = 2\overline{\phi_E}\nabla\phi_E\cdot\nabla\phi_E, \tag{4.40}$$

where the bar indicates complex conjugation and $\nabla$ and $\Delta$ should be interpreted as the ordinary gradient and Laplacian in three dimensions, where we see $(\rho,\varphi,z)$ as cylindrical coordinates. Note that one can also conjugate everything and work with $\overline{\phi_E}$ instead. For the conformal potential $\widetilde{\phi}_E = \Omega^{-\frac{1}{2}}\phi_E$, the Ernst equation gives [37, Eq. (12)]

$$\Big(\widetilde{r}^2\widetilde{\phi}_E\overline{\widetilde{\phi}_E}-1\Big)\widetilde{\Delta}\widetilde{\phi}_E = 2\overline{\widetilde{\phi}_E}\Big(\widetilde{r}^2\widetilde{\nabla}\widetilde{\phi}_E\cdot\widetilde{\nabla}\widetilde{\phi}_E + 2\widetilde{r}\widetilde{\phi}_E\widetilde{\nabla}\widetilde{\phi}_E\cdot\widetilde{\nabla}\widetilde{r} + \widetilde{\phi}_E^2\Big), \tag{4.41}$$

where we now interpret $(\widetilde{\rho},\varphi,\widetilde{z})$ as cylindrical coordinates. To calculate the multipole moments, we also need the Ricci tensor. By hand or using Mathematica, one can check that the only nonvanishing components of the Ricci tensor for $\widetilde{h}$ are

$$\begin{aligned}
\widetilde{R}_{\widetilde{\rho}\widetilde{\rho}} &= \frac{1}{\widetilde{\rho}}\frac{\partial\gamma}{\partial\widetilde{\rho}} - \frac{\partial^2\gamma}{\partial\widetilde{\rho}^2} - \frac{\partial^2\gamma}{\partial\widetilde{z}^2},\\
\widetilde{R}_{\widetilde{z}\widetilde{z}} &= -\frac{1}{\widetilde{\rho}}\frac{\partial\gamma}{\partial\widetilde{\rho}} - \frac{\partial^2\gamma}{\partial\widetilde{\rho}^2} - \frac{\partial^2\gamma}{\partial\widetilde{z}^2},\\
\widetilde{R}_{\widetilde{\rho}\widetilde{z}} = \widetilde{R}_{\widetilde{z}\widetilde{\rho}} &= \frac{1}{\widetilde{\rho}}\frac{\partial\gamma}{\partial\widetilde{z}}.
\end{aligned}$$

But we can write the derivatives of $\gamma$ in terms of $\lambda$ and $w$ by the Einstein equations [112, Eqs. (7.1.24-7.1.27)]

$$\frac{\partial\gamma}{\partial\rho} = \frac{1}{4}\rho\lambda^{-2}\left(\left(\frac{\partial\lambda}{\partial\rho}\right)^2 - \left(\frac{\partial\lambda}{\partial z}\right)^2\right) - \frac{1}{4}\rho^{-1}\lambda^2\left(\left(\frac{\partial w}{\partial\rho}\right)^2 - \left(\frac{\partial w}{\partial z}\right)^2\right), \tag{4.42a}$$

$$\frac{\partial\gamma}{\partial z} = \frac{1}{2}\rho\lambda^{-2}\frac{\partial\lambda}{\partial\rho}\frac{\partial\lambda}{\partial z} - \frac{1}{2}\rho^{-1}\lambda^2\frac{\partial w}{\partial\rho}\frac{\partial w}{\partial z}. \tag{4.42b}$$

Together with the twist one-form which is given by

$$df = \omega = -\frac{1}{\rho}\lambda^2\frac{\partial w}{\partial z}d\rho + \frac{1}{\rho}\lambda^2\frac{\partial w}{\partial\rho}dz,$$

we see that the Ricci tensor can alternatively be written as [37, Eq. (13)]

$$\widetilde{R}_{ij} = \frac{1}{D^2}(G_i\overline{G_j} + \overline{G_i}G_j), \tag{4.43}$$

with $D = \widetilde{r}^2\widetilde{\phi}_E\overline{\widetilde{\phi}_E}-1$, $G_1 = \widetilde{z}\frac{\partial\widetilde{\phi}_E}{\partial\widetilde{\rho}} - \widetilde{\rho}\frac{\partial\widetilde{\phi}_E}{\partial\widetilde{z}}$, $G_2 = \widetilde{\rho}\frac{\partial\widetilde{\phi}_E}{\partial\widetilde{\rho}} + \widetilde{z}\frac{\partial\widetilde{\phi}_E}{\partial\widetilde{z}} + \widetilde{\phi}_E$, $G_3 = 0$.

Now, we are going to make an important assumption. We assume that $\widetilde{\phi}_E$ is analytic. Since $\widetilde{\phi}_E$ is independent of $\phi$, this gives $\widetilde{\phi}_E = \sum_{i,j=0}^{\infty} a_{ij}\widetilde{\rho}^i\widetilde{z}^j$ and only even powers of $\widetilde{\rho}$ can occur.

That is, $a_{ij} = 0$ whenever $i$ is odd. Equation (4.41) constitutes a recursion relation between the coefficients $a_{ij}$ such that they can all be expressed in terms of the constants $a_{0j}$ [37, Eq. (16)]:

$$
\begin{aligned}
(r+2)^2 a_{r+2,s} = &-(s+2)(s+1)a_{r,s+2} \\
&+ \sum_{\substack{k+m+p=r \\ l+n+q=s}} a_{kl}\overline{a_{mn}}\big(a_{pq}(p^2+q^2-4p-5q-2pk-2ql-2) \\
&\qquad + a_{p+2,q-2}(p+2)(p+2-2k) + a_{p-2,q+2}(q+2)(q+1-2l)\big).
\end{aligned}
\tag{4.44}
$$

Equation (4.44) tells us that if we know what happens on the $\widetilde{z}$-axis (i.e., if we know the coefficients $a_{0j}$), then we know $\widetilde{\phi}_E$. In (4.44) and in [37, Eq. (16)], it is not completely clear what the summation boundaries are. If we want to be precise, it is better to write write

$$
\begin{aligned}
(r+s)^2 a_{r+2,s} = &-(s+2)(s+1)a_{r,s+2} \\
&+ \sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q} a_{kl}\overline{a_{mn}}a_{pq}(p^2+q^2-4p-5q-2pk-2ql-2) \\
&+ \sum_{p=-2}^{r}\sum_{q=2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q} a_{kl}\overline{a_{mn}}a_{p+2,q-2}(p+2)(p+2-2k) \\
&+ \sum_{p=2}^{r}\sum_{q=-2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q} a_{kl}\overline{a_{mn}}a_{p-2,q+2}(q+2)(q+1-2l),
\end{aligned}
\tag{4.45}
$$

where $m = r-p-k$ and $n = s-q-l$.

For a symmetric tensor $T$, we let $T_{a,b,c}$ denote the component $T_{1\ldots12\ldots23\ldots3}$ with $a$ 1's, $b$ 2's, and $c$ 3's. The order of the coordinates is $(\widetilde{\rho}, \widetilde{z}, \varphi)$. With the given metric, the Christoffel symbols can easily be calculated and one can check that the recursion relation (4.17) for the multipole moments gives [37, Eq. (18)]

$$
\begin{aligned}
P^n_{a,b,c} = \frac{1}{n}\Bigg(& a\frac{\partial}{\partial\rho}P^{n-1}_{a-1,b,c} + b\frac{\partial}{\partial z}P^{n-1}_{a,b-1,c} \\
&- \left((a(a-1)+2ab)\frac{\partial\gamma}{\partial\widetilde{\rho}} + 2ac\frac{1}{\widetilde{\rho}}\right)P^{n-1}_{a-1,b,c} - (2ab+b(b-1))\frac{\partial\gamma}{\partial\widetilde{z}}(P^A_{n-1})_{a,b-1,c} \\
&+ a(a-1)\frac{\partial\gamma}{\partial\widetilde{z}}(P^A_{n-1})_{a-2,b+1,c} + b(b-1)\frac{\partial\gamma}{\partial\widetilde{\rho}}P^{n-1}_{a+1,b-2,c} \\
&+ c(c-1)\rho e^{-2\gamma}P^{n-1}_{a+1,b,c-2} - \frac{1}{2}(2n-3)\Big(a(a-1)\widetilde{R}_{11}P^{n-2}_{a-2,b,c} \\
&+ 2ab\widetilde{R}_{12}P^{n-2}_{a-1,b-1,c} + b(b-1)\widetilde{R}_{22}P^{n-2}_{a,b-2,c}\Big)\Bigg)^{STF}.
\end{aligned}
\tag{4.46}
$$

Before we take the symmetric trace-free part, we can add terms of the form $\widetilde{h}_{(i_1 i_2}Q_{i_3\ldots i_k)}$ because they vanish when taking the trace-free part by Lemma 4.6. We want to do this in a clever way. Define a collection of symmetric tensors $\left(S^k\right)_{k\in\mathbb{N}_0}$ by $S^0_{0,0,0} = P^0_{0,0,0}$, $S^1_{a,b,c} = P^1_{a,b,c}$

and the recursion relation

$$\begin{aligned}
S^n_{a,b,c} = \frac{1}{n}\bigg( & a\frac{\partial}{\partial\rho}S^{n-1}_{a-1,b,c} + b\frac{\partial}{\partial z}S^{n-1}_{a,b-1,c} \\
& - \left((a(a-1)+2ab)\frac{\partial\gamma}{\partial\widetilde{\rho}} + 2ac\frac{1}{\widetilde{\rho}}\right)S^{n-1}_{a-1,b,c} - (2ab+b(b-1))\frac{\partial\gamma}{\partial\widetilde{z}}S^{n-1}_{a,b-1,c} \\
& + a(a-1)\frac{\partial\gamma}{\partial\widetilde{z}}S^{n-1}_{a-2,b+1,c} + b(b-1)\frac{\partial\gamma}{\partial\widetilde{\rho}}S^{n-1}_{a+1,b-2,c} \\
& + c(c-1)\rho e^{-2\gamma}S^{n-1}_{a+1,b,c-2} - \frac{1}{2}(2n-3)\Big(a(a-1)\widetilde{R}_{11}S^{n-2}_{a-2,b,c} \\
& +2ab\widetilde{R}_{12}S^{n-2}_{a-1,b-1,c} + b(b-1)\widetilde{R}_{22}S^{n-2}_{a,b-2,c}\Big)\bigg) + \widetilde{h}_{(11}Q^{n-2}_{1\ldots12\ldots23\ldots3)}.
\end{aligned}$$

Note that this is just (4.46), except that we do not take trace-free parts anymore and we even add an extra trace parts. The indices in the last term are again such that there are $a$ 1's, $b$ 2's and $c$ 3's. Then the multipole moments are still found by taking the symmetric trace-free part:

$$P^n_{a,b,c} = (S^n)^{STF}_{a,b,c}.$$

We want to show that we can pick $Q^{n-2}_{i_3\ldots i_n}$ such that $S^n_{a,b,c} = 0$ if $c \neq 0$. For $n = 0$, there is nothing to check and for $n = 1$, we observe that the gravitational potentials are independent of $\varphi$, yielding the result immediately. Suppose that we found $Q^{k-2}_{i_3\ldots i_k}$ such that $S^k_{a,b,c} = 0$ if $c \neq 0$ for $k < n$. We take $Q^{n-2}$ such that $Q^{n-2}_{a,b,c} = 0$ when $c \neq 0$, then $S^n_{a,b,c}$ can only be nonzero if $c = 0$ or $c = 2$. If we take

$$Q^{n-2}_{a,b,0} = -\frac{n-1}{\widetilde{\rho}}e^{-2\gamma}S^{n-1}_{a+1,b,0},$$

we indeed find $S^n_{a,b,c} = 0$ if $c \neq 0$ [37, Eq. (22)].[9] So, we can always take $Q^{n-2}$ such that $S^n_{a,b,c} = 0$ if $c \neq 0$ by induction. Write $S^n_a = S^n_{a,n-a,0}$, then the tensors $S^n$ are recursively defined as

$$S^0_0 = \widetilde{\phi}_E, \qquad S^1_0 = \frac{\partial S^0_0}{\partial\widetilde{z}}, \qquad S^1_1 = \frac{\partial S^0_0}{\partial\widetilde{\rho}}, \tag{4.47}$$

and [37, Eq. (23)]

$$\begin{aligned}
S^n_a = \frac{1}{n}\bigg( & a\frac{\partial}{\partial\rho}S^{n-1}_{a-1} + (n-a)\frac{\partial}{\partial z}S^{n-1}_a + a\left((a+1-2n)\frac{\partial\gamma}{\partial\widetilde{\rho}} - \frac{a-1}{\widetilde{\rho}}\right)S^{n-1}_{a-1} \\
& + (a-n)(a+n-1)\frac{\partial\gamma}{\partial\widetilde{z}}S^{n-1}_a + a(a-1)\frac{\partial\gamma}{\partial\widetilde{z}}S^{n-1}_{a-2} \\
& + (n-a)(n-a-1)\left(\frac{\partial\gamma}{\partial\widetilde{\rho}} - \frac{1}{\widetilde{\rho}}\right)S^{n-1}_{a+1} - \frac{1}{2}(2n-3)\Big(a(a-1)\widetilde{R}_{11}S^{n-2}_{a-2} \\
& +2a(n-a)\widetilde{R}_{12}S^{n-2}_{a-1} + (n-a)(n-a-1)\widetilde{R}_{22}S^{n-2}_a\Big)\bigg).
\end{aligned} \tag{4.48}$$

The next step is to show that $S^n_a|_{i^0} = 0$ if $a \neq 0$. We only discuss the strategy how it is done, we do not discuss the proof in full detail. It is possible to formulate the recursion above in terms of $Z^n_a$, where $Z^n_a$ is defined by $S^n_a = \widetilde{\rho}^a Z^n_a$. Then the idea is to count orders in $\widetilde{\rho}$. Using (4.43) and (4.42), it is possible to show that $\frac{1}{\widetilde{\rho}^2}\widetilde{R}_{11}$, $\frac{1}{\widetilde{\rho}}\widetilde{R}_{12}$, $\widetilde{R}_{22}$, $\frac{1}{\widetilde{\rho}}\frac{\partial\gamma}{\partial\widetilde{\rho}}$ and $\frac{1}{\widetilde{\rho}^2}\frac{\partial\gamma}{\partial\widetilde{z}}$ must also be

[9] There is a sign error in [37, Eq. (22)], but it is correct in [37, Eq. (23)].

analytic [37]. Using induction, the recursion relation for $Z_a^n$ shows that $Z_a^n$ is also analytic [37, Eq. (26)]. In particular, $Z_a^n$ is well-defined at $\widetilde{\rho} = 0$ and $S_a^n|_{i^0} = 0$ if $a \neq 0$. So, for the multipole moments we only have to determine the trace-free part of a tensor with only one nonvanishing component, namely $S_0^n|_{i^0} = S_{2\ldots2}^n$. Such a trace-free part has been calculated by Fodor, Hoenselaers and Perjés, which gives [37, Appendix]

$$P_{2\ldots2}^n|_{i^0} = \frac{n!}{(2n-1)!!} S_0^n|_{i^0}.$$

For the constants $C^n$ from (4.38), this gives

$$C^n = \frac{1}{(2n-1)!!} S_0^n. \tag{4.49}$$

To summarise:

**Theorem 4.13** (Fodor–Hoenselaers–Perjés algorithm)**.** *Suppose we have a stationary axisymmetric, asymptotically flat vacuum solution of the Einstein equations and the Ernst potential* $\widetilde{\phi}_E$ *is analytic around* $i^0$*, then the first* $m+1$ *multipole moments can be computed using the following algorithm:*

1. *Find the coefficients* $a_{0j}$ *for* $j \leq m$ *by* $\widetilde{\phi}_E\Big|_{\widetilde{\rho}=0} = \sum_{j=0}^{\infty} a_{0j} \widetilde{z}^j$*;*
2. *Determine* $a_{ij}$ *for* $i + j \leq m$ *using* (4.45)*;*
3. *Calculate the components of the Ricci tensor* $\widetilde{R}_{ij}$ *and the derivatives of* $\gamma$ *in terms of the power series for* $\widetilde{\phi}_E$ *using* (4.42) *and* (4.43)*;*
4. *Compute* $S_a^n$ *for* $n \leq m$ *and* $a \leq m - n$ *using* (4.47) *and* (4.48)*, where we only need to know* $S_a^n$ *up to degree* $\widetilde{\rho}^k \widetilde{z}^l$ *with* $k + l \leq m - n$*;*
5. *Evaluating* (4.49) *for* $n = 0, 1, \ldots, m$ *and using* (4.37)*.*

*Proof.* See the discussion above and the article by Fodor, Hoenselaers and Perjés [37] □

The calculations are still quite heavy. The multipole moments have been expressed in terms of $a_{0j}$ up to order $n = 10$ by Fodor, Hoenselaers and Perjés [37]. In full generality, it is still hard to find a general expression for the multipole moments. However, for a relatively easy solution as the Kerr spacetime, a computer program should be able to calculate the multipole moments to very high order. For the Kerr spacetime in Weyl-Lewis-Papapetrou coordinates, we have [61]

$$\widetilde{\phi}_E(\widetilde{\rho} = 0) = \frac{m}{1 - ia\widetilde{z}} = \sum_{k=0}^{\infty} m(ia)^k \widetilde{z}^k.$$

Therefore, $a_{0j} = m(ia)^j$. One might expect $C^n = a_{0n}$, but that is typically not true. For the Kerr spacetime, however, it is true because $a_{0i}a_{0j} - a_{0,i-1}a_{0,j+1} = 0$. Unfortunately, this is still difficult to prove to arbitrary order. When decomposing the complex Ernst potential into its real and imaginary part, we get the mass and angular momentum multipole moments which turn out to be given by $m_{2k} = (-1)^k ma^{2k}$ and $j_{2k+1} = (-1)^k ma^{2k+1}$ and the other terms vanish.

### Second algorithm to find multipole moments

There is also a more modern method that is often more efficient than the Fodor–Hoenselaers–Perjés algorithm to compute multipole moments due to Bäckdahl and Herberthson [7]. Again, we use Weyl-Lewis-Papapetrou coordinates (4.39). As we discussed in Theorem 3.3 and Proposition 4.10, there is still some freedom in the conformal factor. Therefore, we now take the conformal factor $\Omega = \big(\widetilde{\rho}^2 + \widetilde{z}^2\big)e^{\kappa-\gamma}$ for some smooth function $\kappa$ that vanishes at $i^0$. Then,

$$\widetilde{h} = \Omega^2 h = e^{2\kappa}\big(\widetilde{\rho}^2 e^{-2\gamma} d\varphi^2 + d\widetilde{\rho}^2 + d\widetilde{z}^2\big). \tag{4.50}$$

Like we discussed at the end of Section 4.2, $\kappa$ is typically chosen such that the moments are mass-centered.

We introduce new coordinates $(r, \theta)$ such that $\widetilde{\rho} = r\sin\theta$ and $\widetilde{z} = r\cos\theta$. Then the point $i^0$ corresponds to $r = 0$. Now, we are going to use some magic because we see

$$\eta = \frac{\partial}{\partial\widetilde{z}} - i\frac{\partial}{\partial\widetilde{\rho}} = e^{-i\theta}\left(\frac{\partial}{\partial r} - \frac{i}{r}\frac{\partial}{\partial\theta}\right), \tag{4.51}$$

as a vector field on $\widetilde{S}$, including at $i^0$. At $i^0$, however, the vector field $\eta$ is clearly not well-defined; it is a so-called regularly direction-dependent vector field. In this thesis, we will not discuss such direction-dependent tensors, but they are very useful when working at spatial infinity [4, 52].

The direction-dependent vector field $\eta$ has a few properties.

**Proposition 4.14.** *In $\big(\widetilde{S}, \widetilde{h}\big)$ with $\widetilde{h}$ given by* (4.50)*, the direction-dependent vector field $\eta$ defined by* (4.51)*. Then*

1. *The vector field $\eta$ is a complex null vector field: $\widetilde{h}_{ij}\eta^i\eta^j = 0$;*
2. *For every covariant $k$-tensor field $T$ we have*

$$T^{STF}_{i_1\ldots i_k}\eta^{i_1}\cdots\eta^{i_k} = T_{i_1\ldots i_k}\eta^{i_1}\cdots\eta^{i_k}; \tag{4.52}$$

3. *The covariant derivative $\widetilde{D}_\eta\eta$ is parallel to $\eta$.*

*Proof.* We have

$$\widetilde{h}_{ij}\eta^i\eta^j = e^{2\kappa}\big(1 + (-i)^2\big) = 0,$$

so $\eta$ is indeed a complex null vector field.

By (4.19), we have $T^{STF} = T^S - \big(\widehat{T}\otimes\widetilde{h}\big)^S$ for some symmetric covariant $(k-2)$-tensor field $\widehat{T}$. When fully contracting with $\eta$, we contractions between $\big(\widehat{T}\otimes\widetilde{h}\big)^S$ and $\eta$ vanish because $\widetilde{h}_{ij}\eta^i\eta^j = 0$. Therefore, only the contraction with $T^S$ survives and (4.52) follows.

The last property follows by a direct computation from (4.50). One finds

$$\widetilde{D}_\eta\eta = 2(\mathcal{L}_\eta\kappa)\eta. \tag{4.53}$$

□

We can apply (4.52) to equation (4.37) for the multipole moments in axisymmetric spacetimes. That gives

$$\begin{aligned} P^k\Big|_{i^0}(\eta,\dots,\eta) &= (2k-1)!!C^k\,(d\widetilde{z}\otimes\cdots\otimes d\widetilde{z})^{STF}\Big|_{i^0}(\eta,\dots,\eta) \\ &= (2k-1)!!C^k(d\widetilde{z}(\eta))^k = (2k-1)!!C^k. \end{aligned} \tag{4.54}$$

Therefore, (4.38) tells us that the multipole moments are determined by their contractions with $\eta$.

Equation (4.54) gives an identity at $i^0$, but we define functions $f_k$ on $\widetilde{S}$ by

$$f_k = P^k(\eta,\dots,\eta). \tag{4.55}$$

So, we have $f^k(i^0) = (2k-1)!!C^k$ by (4.54). Moreover, the functions $f_n$ only depend on $\widetilde{\rho}$ and $\widetilde{z}$. We can also contract the recursion relation for the multipole moments with $\eta$. Contracting (4.17) gives, using (4.52),

$$\begin{aligned} f_{k+1} &= \eta^{i_1}\cdots\eta^{i_{k+1}}\Big(\widetilde{D}_{i_{k+1}}P^k_{i_1\dots i_k}\Big) - \frac{1}{2}k(2k-1)\eta^i\eta^j\widetilde{R}_{ij}f_{k-1} \\ &= \widetilde{D}_\eta f_k - P^k_{i_1\dots i_k}\widetilde{D}_\eta\big(\eta^{i_1}\cdots\eta^{i_k}\big) - \frac{1}{2}k(2k-1)\eta^i\eta^j\widetilde{R}_{ij}f_{k-1} \\ &= \widetilde{D}_\eta f_k - 2k(\mathcal{L}_\eta\kappa)f_k - \frac{1}{2}k(2k-1)\eta^i\eta^j\widetilde{R}_{ij}f_{k-1}, \end{aligned} \tag{4.56}$$

where we used (4.53) in the last equality. From the metric (4.50), one can easily find by hand or using Mathematica that

$$\widetilde{Rc}(\eta,\eta) = \widetilde{D}_\eta\Big(\widetilde{D}_\eta\gamma\Big) - \Big(\widetilde{D}_\eta\gamma\Big)^2 - \frac{2i}{\widetilde{\rho}}\widetilde{D}_\eta\gamma - \widetilde{D}_\eta\Big(\widetilde{D}_\eta\kappa\Big) + \Big(\widetilde{D}_\eta\kappa\Big)^2.$$

So, this expresses the recursion for the multipole moments in terms of functions that depend on two variables, namely $\widetilde{\rho}$ and $\widetilde{z}$.

The next step is to go from a recursion on functions of two variables to function on one variable. However, we need some analyticity assumptions again. Suppose $\gamma$ and $\widetilde{\phi}_A$ are analytic around $i^0$ for $A = M, J$. Then $\gamma$ must pick up a factor $\widetilde{\rho}^2$ [7, Lemma 3], from which we see that $\frac{2i}{\widetilde{\rho}}\widetilde{D}_\eta\beta$ is analytic. Therefore, by induction, each $f_n$ is analytic.

Somehow, we want to reduce the functions $f_n$ to functions of one variable. We do this by picking the leading order.

**Definition 4.15.** Suppose $g\colon \mathbb{R}^2 \to \mathcal{C}$ is an analytic function around the origin. Then the *leading order part of* $g$ is the function $g_L\colon \mathbb{R}\to\mathbb{C}$ defined by $g_L(x) = g(x,-ix)$.

The analyticity assumption allows us to write

$$g(\widetilde{z},\widetilde{\rho}) = \sum_{k,l=0}^{\infty} a_{kl}\widetilde{z}^i\widetilde{\rho}^j,$$

on a disk $B_\delta(0)$ for some $\delta > 0$. Using this power series, the definition $g_L$ makes sense on $B_{\delta/\sqrt{2}}(0)$

**Lemma 4.16.** *Let $g$ be an analytic function around the origin with leading order function $g_L$, then $\left(\widetilde{D}_\eta g\right)_L = g'$.*

*Proof.* This is a simply application of the chain rule. We have

$$g_L'(x) = \frac{d}{dx} g(x, -ix) = \frac{\partial g}{\partial \widetilde{z}}(x, -ix) - i\frac{\partial g}{\partial \widetilde{\rho}}(x, -ix) = \left(\widetilde{D}_\eta g\right)_L(x).$$

□

Now, we want to take the leading order part of (4.56). Note that we see $f_n$ as a function of $(\widetilde{z}, \widetilde{\rho})$ (with the coordinates in this order). By the same argument as in the Fodor–Hoenselaers–Perjés algorithm, $f_n$ can only contain powers of $\widetilde{z}$ and $\widetilde{\rho}^2$. Therefore, $f$ is even in $\widetilde{\rho}$ and extend its definition by $f_n(\widetilde{z}, -\widetilde{\rho}) = f_n(\widetilde{z}, \widetilde{\rho})$ if $\widetilde{\rho} > 0$. Hence, we can apply Definition 4.15 to $f_n$.

Let $y_n = (f_n)_L$, then

$$y_{k+1} = y_k' - 2k\kappa_L' y_k - \frac{1}{2}k(2k-1)F y_{k-1}. \tag{4.57}$$

with

$$F = \left(\widetilde{Rc}(\eta, \eta)\right)_L = \gamma_L'' - \left(\gamma_L'\right)^2 + \frac{2}{x}\gamma_L' - \kappa_L'' + \left(\kappa_L'\right)^2.$$

Moreover, we have

$$y_k(0) = f_k(0, 0) = (2k-1)!!C^k.$$

There is still some freedom in the function $\kappa$. If we manage to set $F = 0$, the recursion relation (4.57) simplies greatly because $y_{k+1}$ only depends on $y_k$ and not on $y_{k-1}$ anymore. Take $\kappa$ such that

$$\kappa_L(x) = -\log\left(1 - x\int_0^x \frac{e^{2\gamma_L(u)} - 1}{u^2} du - xC\right) + \gamma_L(x), \tag{4.58}$$

for some constant $C$. Then we have

$$\kappa_L'(x) = \frac{\int_0^x \frac{e^{2\gamma_L(u)}-1}{u^2} du + \frac{e^{2\gamma_L(x)}-1}{x} + C}{1 - x\int_0^x \frac{e^{2\gamma_L(u)}-1}{u^2} du - xC} + \gamma_L'(x),$$

and

$$\kappa_L''(x) = \frac{\frac{2\gamma_L'(x)e^{2\gamma_L(x)}}{x}}{1 - x\int_0^x \frac{e^{2\gamma_L(u)}-1}{u^2} du - xC} + \frac{\left(\int_0^x \frac{e^{2\gamma_L(u)}-1}{u^2} du + \frac{e^{2\gamma_L(x)}-1}{x} + C\right)^2}{\left(1 - x\int_0^x \frac{e^{2\gamma_L(u)}-1}{u^2} du - xC\right)^2} + \gamma_L''(x).$$

After some easy manipulations we find indeed that $F = 0$. Moreover, $\kappa_L'(0) = C$. This leaves the freedom to adapt $C$ such that the multipole moments are, for example, mass-centered. Let $z_n(x) = e^{-2n\kappa_L(x)} y_n(x)$. Then $z_n(0) = y_n(0)$ and (4.57) in terms of $z_n$ becomes

$$z_{k+1}(x) = e^{-2\kappa_L(x)} z_k'(x).$$

Note that the expression for $\kappa_L'(x)$ can be rephrased as

$$x\big(\kappa_L'(x) - \gamma_L'(x)\big) + 1 = e^{\kappa_L(x) + \gamma_L(x)}.$$

The last step is to introduce a new coordinate. Let

$$u = xe^{\kappa_L(x)-\gamma_L(x)}, \tag{4.59}$$

then

$$\frac{du}{dx} = e^{\kappa_L(x)-\gamma_L(x)} + x\big(\kappa_L'(x) - \gamma_L'(x)\big)e^{\kappa_L(x)-\gamma_L(x)} = e^{\kappa_L(x)+\gamma_L(x)}e^{\kappa_L(x)-\gamma_L(x)} = e^{2\kappa_L(x)}.$$

Therefore,

$$z_{k+1}(u) = z_k'(u) = \frac{d^n z_0}{du^n}(u).$$

Thus, the multipole moments can be found by

$$C^k = \frac{1}{(2k-1)!!}y_k(0) = \frac{1}{(2k-1)!!}\frac{d^n z_0}{du^n}(u) = \frac{1}{(2k-1)!!}\frac{d^n y_0}{du^n}(u). \tag{4.60}$$

Let us summarise it again in the following theorem.

**Theorem 4.17** (Backdähl–Herberthson algorithm)**.** *Suppose* $\big(\widetilde{S}, \widetilde{h}\big)$ *is given by* (4.50) *where* $\gamma$ *is analytic and suppose* $\widetilde{\phi}$ *is an analytic function on* $\widetilde{S}$*. Then the multipole moments can be calculated using the following algorithm:*

1. *Calculate* $\gamma_L$ *and, subsequently,* $\kappa_L$ *by* (4.58)*;*
2. *Calculate* $y_0 = \widetilde{\phi}_L$*;*
3. *Introduce the coordinate* $u$ *by* (4.59)*;*
4. *Determine the multipole moments by* (4.60) *and* (4.37)*.*

*Proof.* See the discussion above and the article by Bäckdahl and Herberthson [7]. □

Using this quite simple algorithm, we are finally able to compute the multipole moments for the Kerr spacetime to arbitrary order. We do it in the form of a theorem.

**Theorem 4.18.** *The only nonvanishing multipole moments of the Kerr spacetime are*

$$m_{2k} = (-1)^k ma^{2k}, \qquad j_{2k+1} = (-1)^k ma^{2k+1},$$

*for* $k \in \mathbb{N}_0$*. Here,* $m_k$ *and* $j^k$ *are the constant* $C^k$ *in* (4.38) *for the mass and angular momentum potential, respectively. If we work with the complex potential* $\phi_C = \phi_M + i\phi_J$*, they can nicely be written as*

$$c_k = m(ia)^k,$$

*where* $c_k$ *is the constant* $C^k$ *in* (4.38) *for this complex potential.*

*Proof.* In the naive method, we used the conformal metric (4.29) on $\widetilde{S}$ given by

$$\widetilde{h} = \Omega^2 h = d\overline{R}^2 + \overline{R}^2 d\theta^2 + \frac{\overline{R}^2}{1 - \frac{a^2\overline{R}^2\sin^2\theta}{(1-\frac{1}{4}(m^2-a^2)\overline{R}^2)^2}}\sin^2\theta d\varphi^2.$$

Let $\widetilde{z} = \overline{R}\cos\theta$ and $\widetilde{\rho} = \overline{R}\sin\theta$, then

$$\widetilde{h} = d\widetilde{\rho}^2 + d\widetilde{z}^2 + \widetilde{\rho}^2 e^{-2\gamma} d\varphi^2,$$

with

$$\gamma(\widetilde{z},\widetilde{\rho}) = \frac{1}{2}\log\left(1 - \frac{a^2\widetilde{\rho}^2}{\left(1 - \frac{1}{4}(m^2-a^2)(\widetilde{\rho}^2+\widetilde{z}^2)\right)^2}\right).$$

This is of the wanted form and we can work through the Bäckdahl–Herberthson algorithm, even though we did not derive it from the Kerr spacetime in Weyl-Lewis-Papapetrou coordinates. The gravitational field potentials (4.30) and (4.31) become

$$\widetilde{\phi}_M(\widetilde{z},\widetilde{\rho}) = \frac{m\left(1 + \frac{1}{4}(m^2-a^2)\left(\widetilde{\rho}^2+\widetilde{z}^2\right)\right)}{\left(\left(1 - \frac{1}{4}(m^2-a^2)(\widetilde{\rho}^2+\widetilde{z}^2)\right)^2 - a^2\widetilde{\rho}^2\right)^{\frac{3}{4}}},$$

and

$$\widetilde{\phi}_J(\widetilde{z},\widetilde{\rho}) = \frac{ma\widetilde{z}}{\left(\left(1 - \frac{1}{4}(m^2-a^2)(\widetilde{\rho}^2+\widetilde{z}^2)\right)^2 - a^2\widetilde{\rho}^2\right)^{\frac{3}{4}}}.$$

It is easiest to combine them into the complex potential

$$\widetilde{\phi}_C = \widetilde{\phi}_M + i\widetilde{\phi}_J = \frac{m\left(1 + \frac{1}{4}(m^2-a^2)\left(\widetilde{\rho}^2+\widetilde{z}^2\right)\right) + ima\widetilde{z}}{\left(\left(1 - \frac{1}{4}(m^2-a^2)(\widetilde{\rho}^2+\widetilde{z}^2)\right)^2 - a^2\widetilde{\rho}^2\right)^{\frac{3}{4}}}.$$

The leading order part of $\gamma$ is

$$\gamma_L(x) = \frac{1}{2}\log\left(1 + a^2x^2\right).$$

We want to change the conformal factor with a suitable $\kappa$ as in the discussion above. In equation (4.58), there is the constant $C$. By the naive method to calculate the multipole moments, we already saw that the mass monopole moment is nonvanishing and the mass dipole moment vanishes. Therefore, the moments are mass-centered and we want $\kappa'(0) = 0$. Hence, we take $C = 0$. Then, (4.58) yields

$$\kappa_L(x) = \frac{1}{2}\log\left(1 + a^2x^2\right) - \log\left(1 - a^2x^2\right) = -\frac{1}{2}\log\left(\frac{\left(1 - a^2x^2\right)^2}{1 + a^2x^2}\right).$$

We have to change the conformal factor $\Omega$ correspondingly. The new conformal factor becomes $\widetilde{\Omega} = e^{\kappa}\Omega$. For the leading order part of $\widetilde{\phi}_C$ with the old conformal factor, one easily calculates

$$\left(\widetilde{\phi}_C\right)_L(x) = \frac{m(1 + iax)}{(1 + a^2x^2)^{\frac{3}{4}}}.$$

If we change the conformal factor, we get

$$y_0(x) = e^{-\kappa_L/2}\left(\widetilde{\phi}_C\right)_L(x) = \frac{m(1 + iax)\sqrt{1 - a^2x^2}}{1 - a^2x^2} = \frac{m\sqrt{1 - a^2x^2}}{1 - iax}.$$

For the coordinate $u$, we have

$$u = xe^{\kappa_L(x)-\gamma_L(x)} = \frac{x}{1-a^2x^2}.$$

It is easy to verify that

$$y_0(u) = \frac{m}{\sqrt{1-2iau}},$$

under this coordinate transformation. It is possible to expand this in a power expansion [7]

$$y(u) = m\sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!}(iau)^k.$$

From here, we conclude that

$$c_k = m(ia)^k.$$

Taking the real and imaginary parts gives mass and angular multipole moments, respectively, and we find that the only nonvanishing multipole moments are given by

$$m_{2k} = m(-1)^k a^{2k}, \qquad j_{2k+1} = m(-1)^k a^{2k+1}.$$

□

# Chapter 5

# Thorne formalism

A coordinate-based way to define multipole moments is due to Thorne [107]. The idea is to expand the metric into spherical harmonics, for which we need some suitable coordinates. Then the multipole moments appear as the coefficients in the same way as we are used to in electromagnetism and Newtonian gravity. Since the Thorne formalism heavily relies on spherical harmonics, we start with discussing spherical harmonics in Section 5.1. Then we introduce suitable coordinates and show how the metric decomposes into spherical harmonics in Section 5.2. This defines the multipole moments. We conclude this chapter by calculating the multipole moments for the Kerr solution using the Thorne formalism.

In this section, we will mostly use the same notation as in Thorne [107]: any repeated index will be summed over (also if they are both lower) and all contractions are with respect to the flat metric. The radial vector is represented by $\mathrm{x}$ with length $r$ and the unit radial vector is $\mathrm{n} = \frac{1}{r}\mathrm{x}$ with components $n_j = \frac{x_j}{r}$. For tensors with many indices, we use a multi-index. The multi-index $A_l$ is $(a_1 \dots a_l)$ and $S_{A_l}$ should be read as the $S_{a_1 \dots a_l}$-component of a tensor $S$, but $N_{A_l} = n_{a_1} \cdots n_{a_l}$ for the normal vector.

## 5.1 Spherical harmonics

We want to decompose a covariant 2-tensor field into spherical harmonics. It is not sufficient to consider scalar spherical harmonics, but we also need vector and tensor spherical harmonics. They will all be introduced in this section in this order, for which we follow Thorne [107].

### Scalar spherical harmonics

We start with scalar spherical harmonics. They arise when trying to solve the Laplace equation in spherical coordinates and applying separation of variables. Equivalently, they appear when finding eigenstates of the orbital angular momentum operators $L_z$ and $L^2$.

**Definition 5.1.** The *Legendre polynomials* are given by

$$P_l(x) = {}_2F_1\left(\begin{matrix}-l, l+1\\ 1\end{matrix}; \frac{1-x}{2}\right) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l,$$

for $l \in \mathbb{N}_0$, and the *associated Legendre polynomials* are defined by

$$P_l^m(x) = (-1)^m \big(1-x^2\big)^{m/2} \frac{d^m}{dx^m} P_l(x) = \frac{(-1)^m}{2^l l!} \big(1-x^2\big)^{m/2} \frac{d^{l+m}}{dx^{l+m}} (x^2-1)^l,$$

for $m \in \{-l, -l+1, \ldots, l\}$, where only the second expression defines them for negative $m$. The *scalar spherical harmonics* are defined by

$$Y^{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} e^{im\varphi} P_l^m(\cos\theta).$$

Working out the derivative gives

$$P_l^m(x) = (-1)^m (1-x^2)^{m/2} \sum_{j=0}^{\lfloor \frac{l-m}{2} \rfloor} \frac{(-1)^j}{2^l j!(l-j)!} \frac{(2l-2j)!}{(l-m-2j)!} x^{l-m-2j}.$$

For the scalar spherical harmonics, we use the polar angle $\theta$ and the expression above shows that

$$P_l^m(\cos\theta) = (-1)^m \sin^m\theta \sum_{j=0}^{\lfloor \frac{l-m}{2} \rfloor} \frac{(-1)^j}{2^l j!(l-j)!} \frac{(2l-2j)!}{(l-m-2j)!} \cos^{l-m-2j}\theta.$$

The spherical harmonics satisfy some useful properties. For example, they are eigenfunctions of the spherical Laplacian. Let $\mathrm{L} = -i\mathrm{x} \times \nabla$, then

$$\mathrm{L}^2 f = -\Delta_{\mathbb{S}^2} f = -\left( \frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \left( \sin\theta \frac{\partial f}{\partial\theta} \right) + \frac{1}{\sin^2\theta} \frac{\partial^2 f}{\partial\varphi^2} \right).$$

Together with some other properties, we have the following proposition:

**Proposition 5.2.** *The scalar spherical harmonics*

*(a) are eigenfunctions of* $\mathrm{L}^2$*:*

$$\mathrm{L}^2 Y^{lm} = l(l+1) Y^{lm};$$

*(b) are orthonormal:*

$$\int_0^{2\pi} \int_0^{\pi} Y^{lm}(\theta,\varphi) \overline{Y^{l'm'}}(\theta,\varphi) \sin\theta d\theta d\varphi = \delta_{ll'} \delta_{mm'};$$

*(c) have parity* $\pi = (-1)^l$*:*

$$Y^{lm}(\pi - \theta, \varphi + \pi) = (-1)^l Y^{lm}(\theta, \varphi);$$

*(d) transform under complex conjugation as*

$$\overline{Y^{lm}} = (-1)^m Y^{l,-m}.$$

*Proof.* See, for instance, [45, 56, 91, 107]. □

Alternatively, we can also use symmetric trace-free tensors of rank $l$. In Cartesian coordinates, the unit normal vector is given by $n_x + in_y = e^{i\varphi}\sin\theta$ and $n_z = \cos\theta$. So, we get

$$\begin{aligned} Y^{lm}(\theta,\varphi) &= (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}(n_x+in_y)^m \sum_{j=0}^{\lfloor\frac{l-m}{2}\rfloor} \frac{(-1)^j}{2^l j!(l-j)!}\frac{(2l-2j)!}{(l-m-2j)!}(n_z)^{l-m-2j} \\ &= \mathcal{Y}^{lm}_{A_l} N_{A_l}, \end{aligned} \tag{5.1}$$

where $\mathcal{Y}^{lm}_{k_1\dots k_l}$ is given by

$$\mathcal{Y}^{lm}_{k_1\dots k_l} = \sum_{j=0}^{\lfloor\frac{l-m}{2}\rfloor} c^{lmj}\left(\delta^1_{(k_1} + i\delta^2_{(k_1}\right)\cdots\left(\delta^1_{k_m} + i\delta^2_{k_m}\right)\delta^3_{k_{m+1}}\cdots\delta^3_{k_{l-2j}}\delta_{k_{l-2j+1}k_{l-2j+2}}\cdots\delta_{k_{l-1}k_l)}, \tag{5.2}$$

with

$$c^{lmj} = (-1)^m\left(\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}\right)^{1/2}\frac{(-1)^j}{2^l j!(l-j)!}\frac{(2l-2j)!}{(l-m-2j)!}$$

for $m \geq 0$. The parentheses around the indices represents the symmetric part. For $m < 0$, we see that $\mathcal{Y}^{lm}_{k_1\dots k_l} = (-1)^m \overline{\mathcal{Y}^{l,-m}_{k_1\dots k_l}}$. The tensors $\mathcal{Y}^{lm}$ with $-l \leq m \leq l$ not only generate the spherical harmonics of order $l$, but they also form a basis of the symmetric trace-free tensors of rank $l$. This gives a correspondence between spherical harmonics and such tensors.

**Theorem 5.3.** *There is a one-to-one correspondence between the scalar spherical harmonics and the symmetric, trace-free tensors.*

*Proof.* The proof is sketched above, see Poisson and Will [91, Section 1.5] or Thorne [107, Section II.C] for more details. □

A symmetric, trace-free covariant $l$-tensor $\mathcal{F}$ can be written as

$$\mathcal{F} = \mathcal{F}_{A_l} N_{A_l}.$$

Then the components of $\mathcal{F}$ can be expanded as

$$\mathcal{F}_{A_l} = \sum_{m=-l}^{l} F^{lm}\mathcal{Y}^{lm}_{A_l}, \tag{5.3}$$

for some coefficients $F^{lm}$, because the tensors $\mathcal{Y}^{lm}$ form a basis of the space of symmetric trace-free tensors. The components of $\mathcal{F}$ are real if and only if $F^{l,-m} = (-1)^m\overline{F^{lm}}$. The coefficients $F^{lm}$ can be found from $\mathcal{F}$ by [107, Eq. (2.13)]

$$F^{lm} = 4\pi\frac{l!}{(2l+1)!!}\mathcal{F}_{A_l}\overline{\mathcal{Y}^{lm}_{A_l}}. \tag{5.4}$$

It shows that a function on the two-sphere can both be written in a complex-valued expansion of spherical harmonics and in an expansion of symmetric, trace-free tensors, and the coefficients are related by equations (5.3) and (5.4).

Now, we turn on the radial coordinate again and want to solve the Laplace equation. A general solution is of the form

$$F(r,\theta,\varphi)=\sum_{l,m}\Big(F^{lm}r^{-(l+1)}+G^{lm}r^{l}\Big)Y^{lm}(\theta,\varphi).$$

If we assume the field converges at $\infty$, we have $G^{lm}=0$ and in terms of symmetric, trace-free tensor fields, $F$ is of the form

$$F(r,\mathrm{n})=\sum_{l=0}^{\infty}\mathcal{A}_{A_l}\left(\frac{1}{r}\right)_{,A_l},$$

where

$$f_{,A_l}=\frac{\partial}{\partial x^{a_1}}\cdots\frac{\partial}{\partial x^{a_l}}f.$$

### Vector spherical harmonics

We have discussed scalar spherical harmonics, but in general relativity that is not sufficient because the fundamental object, the metric, is a tensor and not a scalar potential. The next step is to decompose three-dimensional vectors using so-called vector spherical harmonics.

**Definition 5.4.** Let $\xi^0=e_z$, $\xi^1=-\frac{1}{\sqrt{2}}(e_x+ie_y)$, and $\xi^{-1}=\frac{1}{\sqrt{2}}(e_x-ie_y)$. Then the *pure-orbital vector spherical harmonics* are given by

$$\mathrm{Y}^{l',lm}(\theta,\varphi)=\sum_{m'=-l'}^{l'}\sum_{m''=-1}^{1}\langle 1l'm''m'\mid lm\rangle\xi^{m''}Y^{l'm'}(\theta,\varphi),$$

for $l'=l-1,l,l+1$. Here, $\langle l''l'm''m'\mid lm\rangle$ are the Clebsch-Gordan coefficients [107].

They are especially useful because they are again (vector-valued) eigenfunctions of the spherical Laplacian, which is easily seen from Proposition 5.2. This proposition translates to the following result for pure-orbital vector spherical harmonics:

**Proposition 5.5.** *The pure-orbital vector spherical harmonics*

*(a) are vector-valued eigenfunctions of* $\mathrm{L}^2$*:*

$$\mathrm{L}^2\mathrm{Y}^{l',lm}=l'(l'+1)\mathrm{Y}^{l',lm};$$

*(b) are orthonormal:*

$$\int_0^{2\pi}\int_0^{\pi}\mathrm{Y}^{l,LM}(\theta,\varphi)\cdot\overline{\mathrm{Y}^{l',L'M'}(\theta,\varphi)}\sin\theta d\theta d\varphi=\delta_{ll'}\delta_{LL'}\delta_{MM'};$$

*(c) have parity* $\pi=(-1)^{l'+1}$*:*[10]

$$\mathrm{Y}^{l',lm}(\pi-\theta,\varphi+\pi)=-(-1)^{l'+1}\mathrm{Y}^{l',lm}(\theta,\varphi);$$

[10] The extra minus sign represents the inversion of sign of the Cartesian basis vectors under parity inversions.

(d) *transform under complex conjugation as*

$$\overline{\mathrm{Y}^{l',lm}} = (-1)^{l'+l+m+1}\mathrm{Y}^{l',l,-m}.$$

*Proof.* See Edmonds [35] or Thorne [107]. □

There are also other types of vector spherical harmonics. The main advantage of the pure-orbital ones is that they are eigenfunctions of the spherical Laplacian, but they are neither purely radial nor purely transverse. Therefore, we want another type of vector spherical harmonics for describing radiation.

**Definition 5.6.** The *pure-spin vector spherical harmonics* are defined by

$$\mathrm{Y}^{E,lm} = \frac{1}{\sqrt{2l+1}}\left(\sqrt{l+1}\mathrm{Y}^{l-1,lm} + \sqrt{l}\mathrm{Y}^{l+1,lm}\right) = \frac{1}{\sqrt{l(l+1)}}r\nabla Y^{lm} = -\mathrm{n}\times\mathrm{Y}^{B,lm}, \tag{5.5a}$$

$$\mathrm{Y}^{B,lm} = i\mathrm{Y}^{l,lm} = \frac{i}{\sqrt{l(l+1)}}\mathrm{L}Y^{lm} = \mathrm{n}\times\mathrm{Y}^{E,lm}, \tag{5.5b}$$

$$\mathrm{Y}^{R,lm} = \frac{1}{\sqrt{2l+1}}\left(\sqrt{l}\mathrm{Y}^{l-1,lm} - \sqrt{l+1}\mathrm{Y}^{l+1,lm}\right) = \mathrm{n}Y^{lm}. \tag{5.5c}$$

Here, $\nabla$ denotes the Euclidean gradient operator and $\mathrm{L} = -i\mathrm{x}\times\nabla$ is the angular momentum operator.

We see that $\mathrm{Y}^{R,lm}$ is purely radial and $\mathrm{Y}^{E,lm}$ and $\mathrm{Y}^{B,lm}$ are purely transverse. This property makes them very useful for describing radiation and defining multipole moments. Again, we have a version of Proposition 5.5, but we have to forget about the property that they are eigenfunctions of the spherical Laplacian.

**Proposition 5.7.** *The pure-spin vector spherical harmonics*

(a) $\mathrm{Y}^{E,lm}$ *and* $\mathrm{Y}^{B,lm}$ *are purely transverse and* $\mathrm{Y}^{R,lm}$ *is purely radial;*

(b) *are orthonormal:*

$$\int_0^{2\pi}\int_0^{\pi}\mathrm{Y}^{J,lm}(\theta,\varphi)\cdot\mathrm{Y}^{J',l'm'*}(\theta,\varphi)\sin\theta d\theta d\varphi = \delta_{JJ'}\delta_{ll'}\delta_{mm'};$$

(c) $\mathrm{Y}^{E,lm}$ *and* $\mathrm{Y}^{R,lm}$ *have parity* $\pi = (-1)^l$ *and* $\mathrm{Y}^{B,lm}$ *has parity* $\pi = (-1)^{l+1}$*;*

(d) *transform under complex conjugation as*

$$\overline{\mathrm{Y}^{J,lm}} = (-1)^m\mathrm{Y}^{J,l,-m}.$$

*Proof.* See Thorne [107]. □

The index $R$ is clear because it indicates the radial direction, and for the transverse parts we have both $E$ and $B$. But $\mathrm{Y}^{E,lm}$ and $\mathrm{Y}^{B,lm}$ have opposite parity, just like electric and magnetic fields. For $\mathrm{Y}^{E,lm}$, we recognise the same parity as for electric multipoles and for $\mathrm{Y}^{B,lm}$ the same parity as for magnetic multipoles [56, Section 9.8]. Therefore, we say $\mathrm{Y}^{E,lm}$ has electric-type parity and $\mathrm{Y}^{B,lm}$ has magnetic-type parity.

Like we discussed for scalar spherical harmonics, the pure-spin vector spherical harmonics can also be expressed in terms of symmetric trace-free tensors. Inserting equation (5.1), we get

$$Y_j^{E,lm} = \sqrt{\frac{l}{l+1}}\Big(\mathcal{Y}^{lm}_{jA_{l-1}} N_{A_{l-1}}\Big)^T, \tag{5.6a}$$

$$Y_j^{B,lm} = \sqrt{\frac{l}{l+1}}\epsilon_{jpq} n_p \mathcal{Y}^{lm}_{qA_{l-1}} N_{A_{l-1}}, \tag{5.6b}$$

$$Y_j^{R,lm} = n_j \mathcal{Y}^{lm}_{A_l} N_{A_l}, \tag{5.6c}$$

where $(\cdot)^T$ means taking the transverse part of the tensor. The transverse part of a covariant $k$-tensor $T$ is given by [107, Section I.C]

$$T^T_{i_1 \dots i_k} = (\delta_{i_1 j_1} - n_{i_1} n_{j_1}) \cdots (\delta_{i_k j_k} - n_{i_k} n_{j_k}) T_{j_1 \dots j_l}.$$

Similarly, we can also express the pure-orbital vector spherical harmonics in terms of symmetric trace-free tensors, giving

$$Y_j^{l-1,lm} = \sqrt{\frac{l}{2l+1}}\mathcal{Y}^{lm}_{jA_{l-1}} N_{A_{l-1}}, \tag{5.7a}$$

$$Y_j^{l,lm} = -i\sqrt{\frac{l}{l+1}}\epsilon_{jpq} n_p \mathcal{Y}^{lm}_{qA_{l-1}} N_{A_{l-1}}, \tag{5.7b}$$

$$Y_j^{l+1,lm} = -\sqrt{\frac{2l+1}{l+1}}\bigg(n_j \mathcal{Y}^{lm}_{A_l} N_{A_l} - \frac{l}{2l+1}\mathcal{Y}^{lm}_{jA_{l-1}} N_{A_{l-1}}\bigg). \tag{5.7c}$$

An arbitrary vector field that only depends on the spherical coordinates can be expanded in terms of pure-orbital or pure-spin vector spherical harmonics. Say

$$\mathrm{V} = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big(E^{lm}\mathrm{Y}^{E,lm} + B^{lm}\mathrm{Y}^{B,lm} + R^{lm}\mathrm{Y}^{R,lm}\Big).$$

Alternatively, we could also use (5.6) to write

$$V_j = \sum_{l=0}^{\infty}\Big(\big(\mathcal{E}_{jA_{l-1}} N_{A_l}\big)^T + \epsilon_{jpq} n_p \mathcal{B}_{qA_{l-1}} N_{A_{l-1}} + n_j \mathcal{R}_{A_l} N_{A_l}\Big),$$

for some symmetric trace-free tensors $\mathcal{E}$, $\mathcal{B}$ and $\mathcal{R}$. Their coefficients must be given by

$$\mathcal{E}_{A_l} = \sqrt{\frac{l}{l+1}} \sum_{m=-l}^{l} E^{lm}\mathcal{Y}^{lm}_{A_l},$$

$$\mathcal{B}_{A_l} = \sqrt{\frac{l}{l+1}} \sum_{m=-l}^{l} B^{lm}\mathcal{Y}^{lm}_{A_l},$$

$$\mathcal{R}_{A_l} = \sum_{m=-l}^{l} R^{lm}\mathcal{Y}^{lm}_{A_l}.$$

The advantage of the pure-orbital vector spherical harmonics becomes apparent when we want to solve the Laplace equation. An arbitrary vector-valued solution is of the form

$$\mathrm{V}(r,\theta,\varphi) = \sum_{l=0}^{\infty}\sum_{l'=l-1}^{l+1}\sum_{m=-l}^{l}\left(F^{l',lm}r^{-(l'+1)} + G^{l',lm}r^{l'}\right)\mathrm{Y}^{l',lm}(\theta,\varphi).$$

If we, again, assume the field converges at infinity, we have $G^{l',lm} = 0$ and in terms of symmetric, trace-free tensors, a general solution is of the form

$$V_j(r,\mathrm{n}) = \sum_{l=1}^{\infty}\left(\mathcal{B}_{jA_{l-1}}\left(\frac{1}{r}\right)_{,A_{l-1}} + \epsilon_{jpq}\mathcal{C}_{qA_{l-1}}\left(\frac{1}{r}\right)_{,pA_{l-1}}\right) + \sum_{l=0}^{\infty}\mathcal{D}_{A_l}\left(\frac{1}{r}\right)_{,jA_l}.$$

### Tensor spherical harmonics

We discussed two different versions of vector spherical harmonics, which both have their own purposes. The last step we need for the $ij$-components of the metric tensor are spherical harmonics for 2-tensors in three dimensions. For tensor spherical harmonics, we can also define both pure-orbital and pure-spin versions. They are due to Mathews [76] and Zerilli [120].

Since the metric is symmetric, we can restrict ourselves to symmetric tensors. Then there are only 6 linearly independent tensors. A suitable basis is given by

$$\mathrm{t}^m = \sum_{m'=-1}^{1}\sum_{m''=-1}^{1}\langle 11m'm'' \mid 2m\rangle \xi^{m'}\otimes\xi^{m''},$$

for $m = -2, -1, 0, 1, 2$, and

$$3^{-1/2}\delta = -\sum_{m'=-1}^{1}\sum_{m''=-1}^{1}\langle 11m'm'' \mid 00\rangle \xi^{m'}\otimes\xi^{m''}.$$

**Definition 5.8.** The *symmetric pure-orbital tensor spherical harmonics* are

$$\mathrm{T}^{2l',lm} = \sum_{m'=-l'}^{l'}\sum_{m''=-2}^{2}\langle l'2m'm'' \mid lm\rangle Y^{l'm'}\mathrm{t}^{m''},$$

for $l' = l-2, l-1, l, l+1, l+2$, and

$$\mathrm{T}^{0l,lm} = -Y^{lm}3^{-1/2}\delta.$$

The pure-orbital tensor spherical harmonics $\mathrm{T}^{1l',lm}$ are the antisymmetric ones, so they are not of interest to us. The following proposition presents its most important properties:

**Proposition 5.9.** *The symmetric pure-orbital tensor spherical harmonics*

*(a) are tensor-valued eigenfunctions of* $\mathrm{L}^2$*:*

$$\mathrm{L}^2\mathrm{T}^{\lambda l',lm} = l'(l'+1)\mathrm{T}^{\lambda l',lm};$$

*(b) are orthonormal:*

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{T}_{ij}^{\lambda l,LM}(\theta,\varphi)\overline{\mathrm{T}_{ij}^{\lambda' l',L'M'}(\theta,\varphi)}\sin\theta d\theta d\varphi = \delta_{\lambda\lambda'}\delta_{ll'}\delta_{LL'}\delta_{MM'};$$

*(c) have parity $\pi = (-1)^{l'}$:*

$$\mathrm{T}^{\lambda l',lm}(\pi-\theta,\varphi+\pi) = (-1)^{l'}\mathrm{T}^{\lambda l',lm}(\pi,\varphi);$$

*(d) transform under complex conjugation as*

$$\overline{\mathrm{T}^{\lambda l',lm}} = (-1)^{l'+l+m}\mathrm{T}^{\lambda l',l,-m}.$$

*Proof.* See Mathews [76] or Thorne [107]. □

Like for vector spherical harmonics, the pure-orbital harmonics are mostly useful when solving the Laplace equation, but not necessarily when describing radiation. This leads to the pure-spin harmonics.

**Definition 5.10.** The *symmetric pure-spin tensor spherical harmonics* are

$$\mathrm{T}^{L0,lm} = Y^{lm}\mathrm{n}\otimes\mathrm{n}, \tag{5.8a}$$
$$\mathrm{T}^{T0,lm} = 2^{-1/2}Y^{lm}(\delta-\mathrm{n}\otimes\mathrm{n}), \tag{5.8b}$$
$$\mathrm{T}^{E1,lm} = \sqrt{2}\Big(\mathrm{n}\otimes\mathrm{Y}^{E,lm}\Big)^S = \Big(-2\mathrm{n}\times\mathrm{T}^{B1,lm}\Big)^S, \tag{5.8c}$$
$$\mathrm{T}^{E2,lm} = \sqrt{\frac{2}{(l-1)(l+2)}}\Big(r\nabla\mathrm{Y}^{E,lm}\Big)^{STT} = \Big(-\mathrm{n}\times\mathrm{T}^{B2,lm}\Big)^S, \tag{5.8d}$$
$$\mathrm{T}^{B1,lm} = \sqrt{2}\Big(\mathrm{n}\otimes\mathrm{Y}^{B,lm}\Big)^S = \Big(2\mathrm{n}\times\mathrm{T}^{B1,lm}\Big)^S, \tag{5.8e}$$
$$\mathrm{T}^{B2,lm} = \sqrt{\frac{2}{(l-1)(l+2)}}\Big(r\nabla\mathrm{Y}^{B,lm}\Big)^{STT} = \Big(\mathrm{n}\times\mathrm{T}^{E2,lm}\Big)^S. \tag{5.8f}$$

Here, $S$ means the symmetric part and $TT$ the transverse traceless part. The transverse traceless part of a 2-tensor field $T$ is

$$T^{TT}_{i_1i_2} = (\delta_{i_1j_1}-n_{i_1}n_{j_1})(\delta_{i_2j_2}-n_{i_2}n_{j_2})T_{j_1j_2} - \frac{1}{2}(\delta_{i_1i_2}-n_{i_1}n_{i_2})(\delta_{j_1j_2}-n_{j_1}n_{j_2})T_{j_1j_2}.$$

Like we have done in (5.5a) for vector spherical harmonics, one can express the pure-orbital tensor spherical harmonics and pure-spin tensor spherical harmonics in terms of one another. We will not present the relations here; they can be found in Thorne [107, Equations (2.30) and (2.33)].

**Proposition 5.11.** *The symmetric pure-spin tensor spherical harmonics satisfy the following properties:*

*(a)* $\mathrm{T}^{L0,lm}$ *is pure longitudinal and* $\mathrm{T}^{T0,lm}$ *is pure transverse;*

*(b) they are orthonormal:*

$$\int_0^{2\pi}\int_0^{\pi} \mathrm{T}_{ij}^{JS,lm}(\theta,\varphi)\overline{\mathrm{T}_{ij}^{J'S',l'm'}(\theta,\varphi)}\sin\theta d\theta d\varphi = \delta_{JJ'}\delta_{SS'}\delta_{ll'}\delta_{mm'};$$

*(c)* $\mathrm{T}^{L0,lm}$, $\mathrm{T}^{T0,lm}$, $\mathrm{T}^{E1,lm}$ *and* $\mathrm{T}^{E2,lm}$ *have parity* $\pi = (-1)^l$, *and* $\mathrm{T}^{B1,lm}$ *and* $\mathrm{T}^{B2,lm}$ *have parity* $\pi = (-1)^{l+1}$*;*

*(d) they transform under complex conjugation as*[11]

$$\overline{\mathrm{T}^{JS,lm}} = (-1)^m \mathrm{T}^{JS,l,-m}.$$

*Proof.* See Thorne [107]. □

Again, we also want to express the tensor spherical harmonics in terms of symmetric, trace-free tensors. Then we have

$$T_{ij}^{L0,lm} = n_i n_j \mathcal{Y}_{A_l}^{lm} N_{A_l}, \tag{5.9a}$$

$$T_{ij}^{T0,lm} = 2^{-1/2}(\delta_{ij} - n_i n_j)\mathcal{Y}_{A_l}^{lm} N_{A_l}, \tag{5.9b}$$

$$T_{ij}^{E1,lm} = \sqrt{\frac{2l}{l+1}}\Big(n_{(i}\mathcal{Y}_{j)A_{l-1}}^{lm} N_{A_{l-1}} - n_i n_j \mathcal{Y}_{A_l}^{lm} N_{A_l}\Big), \tag{5.9c}$$

$$T_{ij}^{B1,lm} = \sqrt{\frac{2l}{l+1}}\Big(n_{(i}\epsilon_{j)pq}^{lm} n_p \mathcal{Y}_{qA_{l-1}}^{lm} N_{A_{l-1}}\Big), \tag{5.9d}$$

$$T_{ij}^{E2,lm} = \sqrt{\frac{2(l-1)l}{(l+1)(l+2)}}\Big(\mathcal{Y}_{ijA_{l-2}}^{lm} N_{A_{l-2}}\Big)^{TT}, \tag{5.9e}$$

$$T_{ij}^{B2,lm} = \sqrt{\frac{2(l-1)l}{(l+1)(l+2)}}\Big(n_p \epsilon_{pq(i}\mathcal{Y}_{j)qA_{l-2}}^{lm} N_{A_{l-2}}\Big)^{TT}. \tag{5.9f}$$

The expression for the pure-orbital tensor spherical harmonics can be found in Thorne [107, Equation (2.40)]. Any symmetric 2-tensor field on the 2-sphere can be expanded in terms of the pure-orbital or pure-spin tensor spherical harmonics. In particular, solutions of the Laplace equation can be written as

$$\mathrm{U}(r,\theta,\varphi) = \sum_{\lambda,l',l,m}\Big(F^{\lambda l',lm} r^{-(l'+1)} + G^{\lambda l',lm} r^{l'}\Big)\mathrm{T}^{\lambda l',lm}.$$

Again, assuming the solutions converge at infinity, in the form of symmetric, trace-free tensors we get

$$U_{ij}(r,\mathrm{n}) = \sum_{l=0}^{\infty}\delta_{ij}\mathcal{E}_{A_l}\left(\frac{1}{r}\right)_{,A_l} + \sum_{l=2}^{\infty}\left(\mathcal{F}_{ijA_{l-2}}\left(\frac{1}{r}\right)_{,A_{l-2}} + \epsilon_{pqi}\mathcal{G}_{jqA_{l-2}}\left(\frac{1}{r}\right)_{,pA_{l-2}}\right)^S$$
$$+ \sum_{l=1}^{\infty}\left(\mathcal{K}_{iA_{l-1}}\left(\frac{1}{r}\right)_{,jA_{l-1}} + \epsilon_{ipq}\mathcal{I}_{qA_{l-1}}\left(\frac{1}{r}\right)_{,jpA_{l-1}}\right)^S + \sum_{l=0}^{\infty}\mathcal{J}_{A_l}\left(\frac{1}{r}\right)_{,jkA_l}.$$

[11] Note there is a typo in Thorne [107, Equation (2.36b)].

## 5.2 Multipole moments

In Thorne's formalism, the idea is to introduce a very special coordinate system and in that coordinate system the multipole moments can simply be read off from the metric. This section consists of two parts. First we define the multipole moments using a very strong condition on the coordinates. In the second part, we see that the condition on the coordinates can be relaxed if we only need the first few multipole moments.

### Definition of Thorne's multipole moments

Like in Section 4.2, we will consider a stationary asymptotically flat spacetime $(M, g)$ with stationary vector field $\xi$, but the we use a more coordinate-based definition of asymptotic flatness and in four dimensions. Recall from Section 2.2 that the observer space $S$ is a Riemannian manifold consisting of the flow lines of $\xi$ and comes with a natural projection $\pi\colon M \to S$.

**Definition 5.12.** A stationary spacetime $(M, g)$ with stationary vector field $\xi$ is called *coordinate-wise asymptotically flat* if there is a bounded closed subset $K \subseteq S$ of the observer space and a diffeomorphism $\Phi\colon M \setminus \pi^{-1}(K) \to \mathbb{R} \times \big(\mathbb{R}^3 \setminus \overline{B}_R(0)\big)$ such that if we understand $\Phi$ as a chart with Cartesian coordinates $\big(x^0 = t, x^1, x^2, x^3\big)$ on the codomain, then $\frac{\partial}{\partial t} = \xi$ and $g_{\alpha\beta}$ admits a convergent power series representation

$$g_{\alpha\beta} = \eta_{\alpha\beta} + \sum_{l=1}^{\infty} \frac{1}{r^l} g^l_{\alpha\beta}, \tag{5.10}$$

where $r = \sqrt{(x^i)^2 + (x^2)^2 + (x^3)^2}$ and $g^l_{\alpha\beta}$ is independent of $r$. In the spherical coordinates $(r, \theta, \varphi)$ corresponding to $\big(x^1, x^2, x^3\big)$, $g^l_{\alpha\beta}$ is only allowed to depend on $\theta$ and $\varphi$.

Harmonic coordinates $(x^\alpha)$ on $(M, g)$ are defined by

$$\Box_g x^\alpha = 0.$$

Equivalently, define the metric density

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-\det g}\, g^{\alpha\beta},$$

then harmonic coordinates in stationary spacetimes are characterised by the stationary harmonic gauge condition

$$\partial_j \mathfrak{g}^{\alpha j} = 0. \tag{5.11}$$

Since $\frac{\partial}{\partial x^0}$ is a Killing vector field, the metric components are independent of $x^0$. In the actual gauge condition, the sum in equation (5.11) over $j$ should also sum over 0, but that term does not contribute here when we keep the first coordinate vector field to be a Killing vector. To get such coordinates, we perform a coordinate transformation of the form

$$y^\alpha = x^\alpha + f^\alpha\big(x^1, x^2, x^3\big),$$

where each $f^\alpha$ can also be written like a convergent power series as in (5.10) (where we replace $\eta_{\alpha\beta}$ by a constant). Equation (5.11) gives a system of four second order partial differential equations, where only $x^1, x^2, x^3$ appear as variables because the original metric is independent of $x^0$ and $x^0$ does not appear in the coordinate transformation of the metric. Since the functions $f^\alpha$ are independent of $x^0$, the coordinate vector field $\frac{\partial}{\partial x^0}$ is left unchanged and is still the Killing vector field. The power series representations of $f^\alpha$ show that (5.10) is still satisfied in the $y^\alpha$-coordinates.

Using these coordinates, we want to find an expression for the metric. In Thorne [107], this is done by expanding the gravitational field $\overline{h}^{\alpha\beta} = \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta}$. The expansion works in the so-called "weak-field near zone", which is defined using some characteristic parameters. The first one being the mass, the second one being a length scale characterising the nonspherical deformations and the last one being a characteristic time scale on which the multipole moments change. The expansion is done in powers of these characteristic quantities and the order of which spherical harmonics appear. An expression can iteratively be found by calculating the first correction and then using Einstein's equation and the harmonic gauge to calculate next terms. Because of stationarity, we can simplify the expression a bit. In particular, no time derivatives of the multipole moments will appear, no logarithmic terms in $r$ will appear, the metric will be time-independent and the characteristic time scale will not appear. The metric tensor $g_{\alpha\beta}$ can also be decomposed in spherical harmonics, and the coefficients are closely related to the ones for $\overline{h}_{\alpha\beta}$. Executing this procedure gives the following result.

**Theorem 5.13.** *Let $(M, g)$ be a stationary, coordinate-wise asymptotically flat vacuum solution of the Einstein equations (without cosmological constant), then the metric in mass-centered harmonic gauge can be written as*

$$g_{00} = -1 + \frac{2\mathcal{M}}{r} - \frac{2\mathcal{M}^2}{r^2} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left( \frac{2(2l-1)!!}{l!} \mathcal{M}_{A_l} N_{A_l} + S_{l-1} \right), \tag{5.12a}$$

$$g_{0j} = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left( -\frac{4l(2l-1)!!}{(l+1)!} \epsilon_{jka_l} \mathcal{J}_{kA_{l-1}} N_{A_l} + S_{l-1} \right), \tag{5.12b}$$

$$g_{ij} = \delta_{ij} \left( 1 + \frac{2\mathcal{M}}{r} \right) + \frac{\mathcal{M}^2}{r^2} (\delta_{ij} + n_i n_j) + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left( \frac{2(2l-1)!!}{l!} \mathcal{M}_{A_l} N_{A_l} \delta_{ij} + S_{l-1} \right). \tag{5.12c}$$

*Here, $S_{l-1}$ is a symbol denoting a quantity that is independent of $r$, and the angular dependence is contained in spherical harmonics of order at most $l-1$.*

*Proof.* A very brief sketch listing the main steps is given above, see Thorne [107, Section X] for the proof. □

In (5.12), we see the coefficients $\mathcal{M}_{A_l}$ and $\mathcal{J}_{A_l}$ appearing. They are the multipole moments. It is not immediately clear the multipole moments are well-defined because there can be more suitable coordinate systems, but we postpone that discussion to Theorem 5.16. Nevertheless, we do already define them.

**Definition 5.14.** *Thorne's mass $2^l$-pole moment* is $\mathcal{M}_{A_l}$ in (5.12a) and *Thorne's angular momentum $2^l$-pole moment* is $\mathcal{J}_{A_l}$ in (5.12b).

The mass is contained in $g_{00}$ component and the current in the $g_{0j}$ components where time and direction are mixed. In $g_{00}$, we use scalar spherical harmonics and in $g_{0j}$ we use vector spherical harmonics for the decomposition. In $g_{ij}$, we need tensor spherical harmonics, which are appearing in the term $\mathcal{M}_{A_l} N_{A_l}$ in equation (5.12c). Since the mass multipole moments can also be read off from $g_{00}$, where we only need scalar spherical harmonics, tensor spherical harmonics are not very important for applying the construction. However, they are important in deriving the form of the metric as we see they are appearing the $g_{ij}$ components.

### ACMC-$N$ coordinates

In Theorem 5.13, we need mass-centered harmonic coordinates that preserve the coordinate-wise asymptotic flatness condition from Definition 5.12. They are usually very hard to find. Luckily, there is a broader class of coordinate systems for which we can still find the multipole moments up to some finite order. This new class of coordinate systems are the so-called ACMC-$N$ coordinates, which stands for asymptotically Cartesian and mass centered to order $N$.

**Definition 5.15.** A coordinate system $\left(x^0 = t, x^1, x^2, x^3\right)$ is *ACMC-N* if and only if the metric components are independent of $t$ and have the following structure when decomposing the metric in spherical harmonics [107, eq. (11.1)]:

$$\begin{aligned} g_{00} &= -1 + \frac{2\mathcal{M}}{r} + \frac{S_0}{r^2} + \sum_{l=2}^{N} \frac{1}{r^{l+1}} \left( \frac{2(2l-1)!!}{l!} \mathcal{M}_{A_l} N_{A_l} + S_{l-1} \right) \\ &\quad + \frac{1}{r^{N+2}} \left( \frac{2(2N+1)!!}{(N+1)!} \mathcal{M}_{A_{N+1}} N_{A_{N+1}} + (\text{poles with } l \neq N+1) \right) \\ &\quad + \left( \text{terms that die out faster than } \frac{1}{r^{N+2}} \right), \end{aligned} \tag{5.13a}$$

$$\begin{aligned} g_{0j} &= \sum_{l=1}^{N} \frac{1}{r^{l+1}} \left( -\frac{4l(2l-1)!!}{(l+1)!} \epsilon_{jka_l} \mathcal{J}_{kA_{l-1}} N_{A_l} + \left( l \text{ pole with parity } \pi = (-1)^l \right) + S_{l-1} \right) \\ &\quad + \frac{1}{r^{N+2}} \Bigg( -\frac{4(N+1)(2N+1)!!}{(N+2)!} \epsilon_{jka_{N+1}} \mathcal{J}_{kA_N} N_{A_{N+1}} \\ &\qquad + \left( N+1 \text{ pole with parity } \pi = (-1)^{N+1} \right) + (\text{poles with } l \neq N+1) \Bigg) \\ &\quad + \left( \text{terms that die out faster than } \frac{1}{r^{N+2}} \right), \end{aligned} \tag{5.13b}$$

$$\begin{aligned} g_{jk} &= \delta_{jk} + \sum_{l=0}^{N} \frac{1}{r^{l+1}} S_l + \frac{1}{r^{N+2}} (\text{any angular dependence}) \\ &\quad + \left( \text{terms that die out faster than } \frac{1}{r^{N+2}} \right). \end{aligned} \tag{5.13c}$$

A coordinate system is *ACMC-$\infty$* if it is ACMC-$N$ for every $N \in \mathbb{N}_0$.

Comparing equations (5.12) and (5.13), we see that the harmonic coordinate system is ACMC-$N$ for any $N$, so it is ACMC-$\infty$. For ACMC-$N$ coordinates, we can read off $\mathcal{M}_{A_l}$ and $\mathcal{J}_{A_l}$ for $l \leq N+1$. So, if we want to know only the first few multipole moments, it suffices to get coordinates giving a metric of the form (5.13) instead of (5.12). More precisely, if we want to know the multipole moments up to order $2^l$, it suffices to find ACMC-$(l-1)$ coordinates.

We need to check that the multipole moments we read off from the metric are independent of the chosen coordinates. Otherwise, Definition 5.14 would not be well-defined. If we manage to show that the multipole moments up to order $N+1$ are independent of the chosen ACMC-$N$ coordinate system, they are also the same for any harmonic asymptotically flat coordinate system. Hence, it is sufficient for well-definedness of the multipole moments to have independence of the ACMC-$N$ coordinate system for any $N$.

**Theorem 5.16.** *The coefficients $\mathcal{M}_{A_l}$ and $\mathcal{J}_{A_l}$ for $l \leq N+1$ in* (5.13) *are independent of the chosen ACMC-$N$ coordinate system.*

*Proof.* We follow the approach by Thorne [107, Section XI.C]. Suppose we have ACMC-$N$ coordinates $(x^\alpha)$ and apply a coordinate transformation

$$y^\alpha = x^\alpha + f^\alpha\big(x^1, x^2, x^3\big),$$

for some functions $f^\alpha$, such that $(y^\alpha)$ is also in the class of ACMC-$N$ coordinate systems. We do not allow $f^\alpha$ to depend on $x^0$ because we want $\frac{\partial}{\partial x^0} = \frac{\partial}{\partial y^0}$. We expand $f_\alpha = \eta_{\alpha\beta} f^\beta$ in powers of $r$, where we know that powers of positive order cannot appear because the asymptotic flat form of the metric must be preserved. So,

$$f_\alpha\big(x^1, x^2, x^3\big) = \sum_{n=-1}^{\infty} \frac{1}{r^{n+1}} f_\alpha^n(\theta, \varphi),$$

for some functions $f_\alpha^n$ that are independent of both time and the radius. Then, we expand $f_0^n$ in the scalar spherical harmonics $Y^{lm}$ where $l_n$ denotes the maximum order of $l$ that appears in $f_0^n$ and we expand $f_j^n$ in the pure-orbital vector harmonics $Y_j^{l+J,lm}$, where $l_{nJ}$ denotes the maximal order. Write

$$f_0^n = \sum_{l=0}^{l_n} f_0^{nl}(\theta, \varphi), \qquad f_j^n = \sum_{J=-1}^{1} \sum_{l=0}^{l_{nJ}} f_j^{nJl}(\theta, \varphi),$$

where $f_0^{nl}$ contains the terms $Y^{lm}$ and $f_j^{nJl}$ the terms $Y_j^{l+J,lm}$ for all possible $m$.

Let $h_{\alpha\beta}(y) = g_{\alpha\beta}(y) - \eta_{\alpha\beta}$, and expand it in the same way as above where we view them as functions(!). So,

$$h_{\alpha\beta}(x) = \sum_{n=0}^{\infty} r^{-(n+1)} h_{\alpha\beta}^n(\theta, \varphi),$$

and

$$h_{\alpha\beta}^n(\theta, \varphi) = \sum_{K,l} h_{\alpha\beta}^{nKl}(\theta, \varphi).$$

Here, we use the scalar harmonics for the 00-component, pure-orbital vector harmonics for the $0j$-components and the pure-orbital tensor harmonics for the $jk$-components. Because we have ACMC-$N$ components, $h_{\alpha\beta}^n$ contains only poles of order at most $2^l$ with $l \leq n$ for $n \leq N$. For low $n$, we even have that $h_{0j}^0 = 0$ and $h_{00}^1$ contains contains monopoles. This follows from the fact that there is no angular momentum monopole moment and no mass dipole moment in Thorne's formalism.

The metric components in the two coordinate systems are related by

$$g_{\mu\nu}(x) = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu} g_{\alpha\beta}(y(x)) = \eta_{\mu\nu} + h_{\mu\nu}(x) + a_{\mu\nu}(x) + b_{\mu\nu}(x),$$

where

$$a_{\mu\nu}(x) = \left(\delta^\alpha{}_\mu + f^\alpha{}_{,\mu}\right)\left(\delta^\beta{}_\nu + f^\beta{}_{,\nu}\right) h_{\alpha\beta}(x + f(x)) - h_{\mu\nu}(x) + f^\alpha{}_{,\mu} f_{\alpha,\nu},$$

and

$$b_{\mu\nu}(x) = f_{\mu,\nu} + f_{\mu,\nu}.$$

Here, the comma notation means the covariant derivative with respect to the flat metric. That means, it is just a partial derivative if we use Cartesian coordinates, but the flat Christoffel symbols do appear in spherical coordinates for example. We also expand $g_{\mu\nu}$, $a_{\mu\nu}$ and $b_{\mu\nu}$ in powers of $\frac{1}{r}$. Because we are working in ACMC-$N$, we see that $g^n_{\mu\nu}$ obeys the same conditions as $h^n_{\mu\nu}$, showing that $a^n_{\mu\nu} + b^n_{\mu\nu}$ also must obey the same rules. That means, $a^0_{0j} + b^0_{0j} = 0$, $a^1_{00} + b^1_{00}$ contains only monopoles and for the other terms $a^n_{\mu\nu} + b^n_{\mu\nu}$ contains only harmonics of order $l \leq n$.

Looking at the expression for $a_{\mu\nu}(x)$, we see that $a^0_{\mu\nu} = 0$, so we have that $b^0_{0j} = 0$ and $b^0_{00}$ and $b^0_{jk}$ are only allowed to contain monopoles. Using the expansion for $f$ we have $b^0_{00} = 0$,

$$b^0_{0j} = \sum_{l=0}^{l_{-1}} r f^{-1l}_{0,j},$$

and

$$b^0_{jk} = \sum_{J=-1}^{1} \sum_{l=0}^{l_{(-1)J}} r(f^{-1Jl}_{j,k} + f^{-1Jl}_{k,j}).$$

The only constant scalar harmonic is with $l = 0$ and therefore $l_{-1} = 0$ in order to fulfill the condition $b^0_{0j} = 0$. For $b^0_{jk}$ there is a bit more freedom and after carefully comparing the spherical harmonics it turns out that we need $l_{(-1)(-1)} = 1$ and $l_{(-1)0} = l_{(-1)1} = 0$. Because $a^0_{00} + b^0_{00} = 0$, we already see that the mass monopole moment $\mathcal{M}$ is the same for both coordinate systems.

As $n$ increases, there are more terms and the situation becomes more delicate. For $n = 1$, we find that $a^1_{00} + b^1_{00}$ can only contain monopoles and $a^1_{0j} + b^1_{0j}$ and $a^1_{jk} + b^1_{jk}$ can contain monopoles and dipoles. Still $b_{00} = 0$, so we are left with

$$a^1_{00} = r^2 \sum_{J=-1}^{1} \sum_{l=0}^{l_{(-1)J}} f^{-1Jl}_j (r^{-1} h^0_{00})_{,j} = -\left(\sum_{J=-1}^{1} f^{-1J0}_j n_j + f^{(-1)(-1)1}_j n_j\right) h^0_{00},$$

where we also used that $h^0_{00}$ is constant. This must be a monopole, but the last term has $l = 1$. This is not allowed and therefore we must reset $l_{(-1)(-1)}$ to zero as well. Repeating the procedure for the other components of $a^1_{\mu\nu} + b^1_{\mu\nu}$ shows that $l_0 = 1$ and $l_{0J} = 1$ [107, Eq. (11.17)]. Since $a^1_{0j} + b^1_{0j}$ only contains harmonics of parity $\pi = (-1)^l$, the current dipole moment is the same in both coordinate systems. Since both systems are mass-centered, the mass dipole moments both vanish.

Up to now, we have shown that $l_{n-1} = l_{(n-1)J} = n$ for $n = 0$ and $n = 1$. By induction we want to show that this holds for all $n \leq N$. Assume it holds for $0, 1, \ldots, n-1$, where $n \leq N$. The term $a^n_{\mu\nu}$ contains basically two types of terms, the first type coming from $f^\alpha{}_\mu f_{\alpha,\nu}$ and second one coming from $h_{\alpha\beta}(x + f(x))$. More precisely, the first type of terms are of the form $f^{p_1 J_1 l_1}_{j_1} f^{p_2 J_2 l_2}_{j_2}$ with $p_1 + p_2 + 3 = n$. This gives harmonics of order $l \leq l_1 + l_2$ and we must have $p_1, p_2 \leq n-2$, so $l \leq n-1$. These terms only contain poles of order at most $n-1$, which is fine. The second type of terms are of the form $f^{p_1 J_1 l_1}_{j_1} \cdots f^{p_k J_k l_k}_{j_k} h^{qKL}$ with $p_1 + \cdots + p_k + 2k + q = n$, where $k \geq 1$. This consists of harmonics of order

$$l \leq l_1 + \cdots + l_k + L \leq l_{p_1 J_1} + \cdots + l_{p_k J_k} + q \leq p_1 + \cdots + p_k + k + q = n - k \leq n - 1,$$

where we used the induction hypothesis to conclude $l_{p_j J_j} \leq p_j + 1$. Similarly as above we have $b^n_{00} = 0$,

$$b^n_{0j} = \sum_{l=0}^{l_{n-1}} r^{n+1} (r^{-n} f_0^{(n-1)l})_{,j},$$

and

$$b^n_{jk} = \sum_{J=-1}^{1} \sum_{l=0}^{l_{(n-1)J}} r^{n+1} ((r^{-n} f_j^{(n-1)Jl})_{,k} + (r^{-n} f_k^{(n-1)J})_{,j}).$$

Because we are using ACMC-$N$ coordinates, $a^n_{\mu\nu} + b^n_{\mu\nu}$ is not allowed to contain poles of order more than $n$. By the form of $a^n_{\mu\nu}$, the same must hold for $b^n_{\mu\nu}$, and this tells us that $l_{n-1} = l_{(n-1)J} = n$. This proves the induction and we see that both coordinate systems must have the same mass and current pole moments of order $n$. For the current moments we use the parity of the terms appearing in $a^n_{\mu\nu} + b^n_{\mu\nu}$ again.

Now, we have proven that the multipole moments are the same for $n \leq N$, but we were also able to define them of order $N+1$ in equation (5.13). Still $a^{N+1}_{\mu\nu}$ only contains harmonics of order at most $N$ by exactly the same reasoning, but the characterization for the orders appearing in $a^{N+1}_{\mu\nu} + b^{N+1}_{\mu\nu}$ does not need to hold anymore. Since the $\frac{1}{r^{N+2}}$-terms in equation (5.13b) also allow for higher order poles and only care about the pole of order $N+1$ and $b^{N+1}_{\mu\nu}$ only contains poles with parity $\pi = (-1)^l$, the current $(N+1)$-pole moment can still be read off. The same holds for the mass $(N+1)$-pole moment because $b^{N+1}_{00} = 0$. Therefore, the multipole moments of order at most $N+1$ are independent of the coordinates when using ACMC-$N$ coordinate systems. □

In equations (5.12) and (5.13), we used the spherical harmonics in terms of the symmetric trace-free tensors. Alternatively, we can also use the scalar and pure-spin vector spherical harmonics, which are related by equations (5.1) and (5.6). Then the metric components in an ACMC-$N$ coordinate system become

$$g_{00} = -1 + \frac{2M}{r} + \frac{S_0}{r^2} + \sum_{l=2}^{N} \frac{1}{r^{l+1}} \left( \frac{(2l-1)!!}{2} \left( \frac{2(l-1)l}{(l+1)(l+2)} \right)^{1/2} \sum_{m=-l}^{l} M^{lm} Y^{lm} + S_{l-1} \right)$$
$$+ \frac{1}{r^{N+2}} \left( \frac{(2N+1)!!}{2} \left( \frac{2N(N+1)}{(N+2)(N+3)} \right)^{1/2} \sum_{m=-(N+1)}^{N+1} M^{(N+1)m} Y^{(N+1)m} \right.$$
$$\left. + (\text{poles with } l \neq N+1) \right)$$
$$+ \left( \text{terms that die out faster than } \frac{1}{r^{N+2}} \right), \tag{5.14a}$$

$$g_{0j} = \frac{1}{r^2} (-2\varepsilon_{jpq} J_p n_q + (1 \text{ pole with parity } \pi = -1))$$
$$+ \sum_{l=2}^{N} \frac{1}{r^{l+1}} \left( -\frac{(2l-1)!!}{2} \left( \frac{2(l-1)}{l+2} \right)^{1/2} \sum_{m=-l}^{l} J^{lm} Y_j^{B,lm} \right.$$
$$\left. + \left( l \text{ pole with parity } \pi = (-1)^l \right) + S_{l-1} \right)$$
$$+ \frac{1}{r^{N+2}} \left( -\frac{(2N+1)!!}{2} \left( \frac{2N}{N+3} \right)^{1/2} \sum_{m=-(N+1)}^{N+1} J^{(N+1)m} Y_j^{B,(N+1)m} \right.$$
$$\left. + \left( N+1 \text{ pole with parity } \pi = (-1)^{N+1} \right) + (\text{poles with } l \neq N+1) \right)$$
$$+ \left( \text{terms that die out faster than } \frac{1}{r^{N+2}} \right), \tag{5.14b}$$

$$g_{jk} = \delta_{jk} + \sum_{l=0}^{N} \frac{1}{r^{l+1}} S_l + \frac{1}{r^{N+2}} (\text{any angular dependence})$$
$$+ \left( \text{terms that die out faster than } r^{-(N+2)} \right). \tag{5.14c}$$

The new multipole moments are $M^{lm}$ and $J^{lm}$ and can again be read off from the metric. The different multipole moments are related to each other by

$$M^{lm} = \frac{16\pi}{(2l+1)!!} \left( \frac{(l+1)(l+2)}{2(l-1)l} \right)^{1/2} \mathcal{M}_{A_l} \overline{\mathcal{Y}_{A_l}^{lm}},$$

and

$$J^{lm} = -\frac{32\pi l}{(l+1)(2l+1)!!} \left( \frac{(l+1)(l+2)}{2(l-1)l} \right)^{1/2} \mathcal{J}_{A_l} \overline{\mathcal{Y}_{A_l}^{lm}}.$$

The inverse relations are given by

$$\mathcal{M}_{A_l} = \frac{l!}{4} \left( \frac{2(l-1)l}{(l+1)(l+2)} \right)^{1/2} \sum_{m=-l}^{l} M^{lm} \mathcal{Y}_{A_l}^{lm}, \tag{5.15a}$$

and

$$\mathcal{J}_{A_l} = -\frac{(l+1)!}{8l}\left(\frac{2(l-1)l}{(l+1)(l+2)}\right)^{1/2}\sum_{m=-l}^{l} J^{lm}\mathcal{Y}^{lm}_{A_l}. \tag{5.15b}$$

In conclusion, we have found two versions of multipole moments by using different versions of the spherical harmonics. The ones in (5.13) are most useful when working with Cartesian coordinates. If we rewrite the components into spherical coordinates, (5.14) is more useful. It does not matter which ones we take, they are equivalent by the relations above.

## 5.3 Kerr spacetime

To illustrate the formalism, we want to calculate the multipole moments for the Kerr spacetime. The difficulty in Thorne's formalism when calculating multipole moments is to find suitable coordinates. With the right coordinates, the multipole moments can be found by just reading off the metric components. To do the procedure for the Kerr spacetime, we follow [107, Section XI.D].

We start with the metric in Boyer–Lindquist coordinates, see equation (4.26). We normalise the coordinate vectors with respect to the ordinary flat metric in spherical coordinates, and expanding the corresponding metric components in powers of $\frac{1}{r}$ gives

$$\begin{aligned}
g_{tt} &= -1 + \frac{2m}{r} - \frac{2ma^2\cos^2\theta}{r^3} + O\left(\frac{1}{r^5}\right),\\
g_{t\varphi} &= -\frac{2ma\sin\theta}{r^2} + \frac{2ma^3\cos^2\theta\sin\theta}{r^4} + O\left(\frac{1}{r^6}\right),\\
g_{rr} &= 1 + \frac{2m}{r} + \frac{4m^2 - a^2\sin^2\theta}{r^2} + \frac{8m^3 - 2ma^2(2-\cos^2\theta)}{r^3} + O\left(\frac{1}{r^4}\right),\\
g_{\theta\theta} &= 1 + \frac{a^2\cos^2\theta}{r^2},\\
g_{\varphi\varphi} &= 1 + \frac{a^2}{r^2} + \frac{2ma^2\sin^2\theta}{r^3} + O\left(\frac{1}{r^5}\right).
\end{aligned}$$

Now, $g_{tt}$ contains no $\frac{1}{r^2}$-term, so the coordinates are mass-centered. However, the $\frac{1}{r^2}$-terms in $g_{rr}$ and $g_{\theta\theta}$ contain second-order spherical harmonics, so we only have ACMC-0 coordinates. From this, we can already read off that $\mathcal{M} = m$, $\mathcal{J} = 0$, $\mathcal{M}_a = 0$ and $\mathcal{J}_\theta = -ma\sin\theta$ and $\mathcal{J}_\varphi = 0$, but it is less clear what to do with $\mathcal{J}_r$. It is determined by the fact that the multipole moments are symmetric and trace-free tensors.

When switching to Cartesian coordinates, it is more easily read off that $\mathcal{J}_z = ma$ and $\mathcal{J}_x = \mathcal{J}_y = 0$. Alternatively, we can use the form of equation (5.14), which give exactly the same results.

Only the monopole and dipole mass and current moments may not give enough information about radiation. We want to calculate the next multipole moments, for which we introduce new coordinates which are of class ACMC-$N$ with $N \geq 1$. Define $r'$ and $\theta'$ by $r = r' + \frac{a^2\cos^2\theta'}{2r'}$ and $\theta = \theta' - \frac{a^2\cos\theta'\sin\theta'}{2r'^2}$. Transforming the metric into these coordinates, normalising it again and expanding in powers of $\frac{1}{r'}$ gives

$$\begin{aligned}
g_{tt} &= -1 + \frac{2m}{r'} - \frac{3ma^2\cos^2\theta'}{r'^3} + O\left(\frac{1}{r'^5}\right) \\
&= -1 + \frac{2m}{r'} - \frac{ma^2(1+2P^2)}{r'^3} + O\left(\frac{1}{r'^5}\right), \\
g_{t\varphi} &= -\frac{2ma\sin\theta'}{r'^2} + \frac{5ma^3\cos^2\theta'\sin\theta'}{r'^4} + O\left(\frac{1}{r'^6}\right) \\
&= -\frac{2ma\sin\theta'}{r'^2} - \frac{ma^3(\partial_\theta P^1 + \frac{2}{3}\partial_\theta P^3)}{r'^4} + O\left(\frac{1}{r'^6}\right), \\
g_{r'r'} &= 1 + \frac{2m}{r'} + \frac{4m^2 - a^2}{r'^2} + \frac{8m^3 - 4ma^2 - ma^2\cos^2\theta'}{r'^3} + O\left(\frac{1}{r'^4}\right), \\
g_{r'\theta'} &= -\frac{2ma^2\cos\theta'\sin\theta'}{r'^4} + O\left(\frac{1}{r'^4}\right), \\
g_{\theta'\theta'} &= 1 + \frac{a^2}{r'^2} + O\left(\frac{1}{r'^4}\right), \\
g_{\varphi\varphi} &= 1 + \frac{a^2}{r'^2} + \frac{2ma^2\sin^2\theta'}{r'^3} + O\left(\frac{1}{r'^4}\right).
\end{aligned}$$

This new coordinate system is of type ACMC-2, allowing us to read off the quadrupole and octopole moments. We easily see that $\mathcal{M}_{a_1a_2a_3} = 0$ and $\mathcal{J}_{a_1a_2} = 0$ since the $\frac{1}{r'^4}$-term is absent in $g_{tt}$ and the $\frac{1}{r'^3}$-term is absent in $g_{t\varphi}$. For the remaining terms, we use the form of equation (5.14). The only nonvanishing quadrupole and octopole moments are $M^{20} = -4\sqrt{\frac{4\pi}{15}}ma^2$ and $J^{30} = \frac{4}{3}\sqrt{\frac{4\pi}{105}}ma^3$. Using equation (5.15), we can also express the multipole moments in terms of symmetric trace-free tensors, where we see that

$$\mathcal{M}_{a_1a_2} = \frac{1}{\sqrt{12}}\left(-4\sqrt{\frac{4\pi}{15}}ma^2\right)\mathcal{Y}^{20}_{a_1a_2} = \frac{1}{3}ma^2\big(\delta_{a_1a_2} - 3\delta^3_{a_1}\delta^3_{a_2}\big),$$

so

$$\mathcal{M}_{xx} = \mathcal{M}_{yy} = \frac{1}{3}ma^2, \quad \mathcal{M}_{zz} = -\frac{2}{3}ma^2,$$

are the nonvanishing components of the quadrupole mass moment. Similarly,

$$\mathcal{J}_{a_1a_2a_3} = \frac{2}{15}ma^3\big(\delta^3_{a_1}\delta_{a_2a_3} + \delta^3_{a_2}\delta_{a_3a_1} + \delta^3_{a_3}\delta_{a_1a_2} - 5\delta^3_{a_1}\delta^3_{a_2}\delta^3_{a_3}\big),$$

so the nonvanishing terms are

$$\mathcal{J}_{xxz} = \mathcal{J}_{xzx} = \mathcal{J}_{zxx} = \mathcal{J}_{yyz} = \mathcal{J}_{yzy} = \mathcal{J}_{zyy} = \frac{2}{15}ma^3, \quad \mathcal{J}_{zzz} = -\frac{4}{15}ma^3.$$

So, we have found the mass and current multipole moments up to poles of order $2^3$. For higher order moments we need coordinates that are of type ACMC-$N$ with $N \geq 3$. In [104], Sopuerta and Yunes mention a coordinate system of class ACMC-6, and it is possible to calculate the multipole moments up to poles of order $2^7$. Ultimately, we would like to have harmonic coordinates to read off the multipole moments of any order. There are several harmonic coordinates for the Kerr spacetime [1, 59, 119]. With these coordinate systems, we read off the multipole moments up to finite order using software like Wolfram Mathematica, but it is still difficult to prove a general formula for any order.

# Chapter 6

# Multipole moments in vacuum

In Chapter 4 and Chapter 5, we defined multipole moments in two very different, seemingly unrelated, ways. The Geroch–Hansen formalism is more geometric and the Thorne formalism is very coordinate-dependent. In this chapter, we investigate how the two versions of multipole moments are related. This question was answered by Gürsel [46] in 1983, who proved that the multipole moments are actually equivalent. The goal of Section 6.1 is to go through Gürsel's work and prove that

$$\mathcal{M}_{a_1\ldots a_k} = \frac{1}{(2k-1)!!} M^k_{a_1\ldots a_k}, \tag{6.1}$$

and

$$\mathcal{J}_{a_1\ldots a_k} = \frac{k+1}{2k(2k-1)!!} J^k_{a_1\ldots a_k}, \tag{6.2}$$

where the Thorne multipole moments on the left-hand side are defined by Definition 5.14 and the Geroch–Hansen multipole moments on the right-hand side are defined by Definition 4.9. In Section 6.2, we derive some properties of the multipole moments.

## 6.1 Equivalence of both formalisms

Before we can prove equations (6.1) and (6.2) in Theorem 6.4, we discuss the assumptions. Since we are only interested in what happens at infinity, we can remove some bounded part of the spacetime. We assume that $S$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B}_R(0)$ for some $R > 0$ such that $M$ is coordinate-wise asymptotically flat according to Definition 5.12 with empty $K$. If necessary, we increase $R$ such that $S$ plays the role of $S \setminus K$ in Theorem 3.3. Then, the Thorne and Geroch–Hansen multipole moments both exist. We also need some lemmas.

**Lemma 6.1.** *Let $(S, h)$ be a three-dimensional Riemannian manifold and let $x = (x^1, x^2, x^3)$ be a harmonic coordinate system globally on $(S, h)$. Let $\alpha$ be a smooth positive function on $S$ and let $\widetilde{h} = \alpha^2 h$. Then a coordinate system $\widetilde{x} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$ is harmonic with respect to $\widetilde{h}$ if and only if $\widetilde{x}^a(x^1, x^2, x^3)$, $a = 1, 2, 3$, are solution of*

$$D_i\big(\alpha D^i \widetilde{x}^a\big) = 0, \tag{6.3}$$

*where $D$ denotes the Levi-Civita connection on $(S, h)$. Moreover,* (6.3) *is equivalent to*

$$h^{ij}\frac{\partial}{\partial x^i}\left(\alpha\frac{\partial\widetilde{x}^a}{\partial x^j}\right) = 0.$$

*Proof.* We follow Gürsel [46, Lemma 1]. Because a metric is a tensor, in $\widetilde{x}$-coordinate $\widetilde{h} = \alpha^2 h$ reads

$$\widetilde{h}_{ab} = \widetilde{h}_{ij}\frac{\partial x^i}{\partial\widetilde{x}^a}\frac{\partial x^j}{\partial\widetilde{x}^b} = \alpha^2 h_{ij}\frac{\partial x^i}{\partial\widetilde{x}^a}\frac{\partial x^j}{\partial\widetilde{x}^b}.$$

For the determinant, this relation gives

$$\det\widetilde{h} = \alpha^6(\det h)\left(\det\frac{\partial\widetilde{x}}{\partial x}\right)^{-2},$$

where $\frac{\partial\widetilde{x}}{\partial x}$ is the matrix with components $\frac{\partial\widetilde{x}^a}{\partial x^i}$. So

$$\begin{aligned}\frac{\partial}{\partial\widetilde{x}^b}\left(\sqrt{\det\widetilde{h}}\widetilde{h}^{ab}\right) &= \frac{\partial}{\partial\widetilde{x}^b}\left(\alpha^3\sqrt{\det h}\left(\det\frac{\partial\widetilde{x}}{\partial x}\right)^{-1}\alpha^{-2}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\frac{\partial\widetilde{x}^b}{\partial x^k}\right)\\ &= \frac{\partial}{\partial\widetilde{x}^b}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\frac{\partial\widetilde{x}^b}{\partial x^k}\left(\det\frac{\partial\widetilde{x}}{\partial x}\right)^{-1}\right).\end{aligned}$$

Using that the adjugate matrix of an invertible matrix is given by its determinant times its inverse, we find

$$\left(\det\frac{\partial\widetilde{x}}{\partial x}\right)^{-1}\frac{\partial\widetilde{x}^b}{\partial x^k} = \frac{1}{2}\delta^{bcd}_{klm}\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d},$$

where $\delta^{bcd}_{klm}$ equals 1 if $bcd$ is an even permutation of $klm$, equals $-1$ if $bcd$ is an odd permutation of $klm$ and equals 0 otherwise. Therefore,

$$\begin{aligned}\frac{\partial}{\partial\widetilde{x}^b}\left(\sqrt{\det\widetilde{h}}\widetilde{h}^{ab}\right) &= \frac{1}{2}\delta^{bcd}_{klm}\frac{\partial}{\partial\widetilde{x}^b}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d}\right)\\ &= \frac{1}{2}\delta^{bcd}_{klm}\frac{\partial}{\partial\widetilde{x}^b}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\right)\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d}\\ &\quad+\frac{1}{2}\delta^{bcd}_{klm}\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\frac{\partial^2 x^l}{\partial\widetilde{x}^b\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d}\\ &\quad+\frac{1}{2}\delta^{bcd}_{klm}\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial^2 x^m}{\partial\widetilde{x}^b\partial\widetilde{x}^d}.\end{aligned}$$

Now, $\frac{\partial^2 x^l}{\partial\widetilde{x}^b\partial\widetilde{x}^c}$ is symmetric in $b$ and $c$ while $\delta^{bcd}_{klm}$ in antisymmetric is $b$ and $c$. Therefore, the contraction vanishes and we see the second term in the expression above vanishes. Similarly, the third term also vanishes and we are left with

$$\begin{aligned}\frac{\partial}{\partial\widetilde{x}^b}\left(\sqrt{\det\widetilde{h}}\widetilde{h}^{ab}\right) &= \frac{1}{2}\delta^{bcd}_{klm}\frac{\partial}{\partial\widetilde{x}^b}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\right)\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d}\\ &= \frac{1}{2}\frac{\partial x^i}{\partial\widetilde{x}^b}\delta^{bcd}_{klm}\frac{\partial x^l}{\partial\widetilde{x}^c}\frac{\partial x^m}{\partial\widetilde{x}^d}\frac{\partial}{\partial x^i}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\right)\\ &= \left(\det\frac{\partial\widetilde{x}}{\partial x}\right)^{-1}\frac{\partial}{\partial x^k}\left(\alpha\sqrt{\det h}h^{jk}\frac{\partial\widetilde{x}^a}{\partial x^j}\right),\end{aligned}$$

where we used that the adjugate matrix is again the determinant times the inverse in the last equality. On the other hand,

$$D_i\big(\alpha D^i\widetilde{x}^a\big) = D_i\left(\alpha h^{ij}\frac{\partial \widetilde{x}^a}{\partial x^j}\right) = \frac{1}{\sqrt{\det h}}\frac{\partial}{\partial x^i}\left(\alpha\sqrt{\det h}h^{ij}\frac{\partial \widetilde{x}^a}{\partial x^j}\right).$$

Therefore,

$$\frac{\partial}{\partial \widetilde{x}^b}\left(\sqrt{\det \widetilde{h}}\widetilde{h}^{ab}\right) = \frac{\sqrt{\det h}}{\det \frac{\partial \widetilde{x}}{\partial x}} D_i\big(\alpha D^i\widetilde{x}^a\big).$$

The coordinates are harmonic if their Laplacians vanish, and this is precisely the Laplacian of $\widetilde{x}^b$ with respect to $\widetilde{h}$ up to a factor. Therefore, $\widetilde{x}$ is a harmonic coordinate system if and only if $D_i\big(\alpha D^i\widetilde{x}^a\big) = 0$. Using that $x$ is a harmonic coordinate system, we can also write

$$D_i\big(\alpha D^i\widetilde{x}^a\big) = \frac{1}{\sqrt{\det h}}\frac{\partial}{\partial x^i}\left(\alpha\sqrt{\det h}h^{ij}\frac{\partial \widetilde{x}^a}{\partial x^j}\right) = h^{ij}\frac{\partial}{\partial x^i}\left(\alpha\frac{\partial \widetilde{x}^a}{\partial x^j}\right),$$

which finishes the proof. □

**Lemma 6.2.** *Let $(S, h)$ be a three-dimensional Riemannian manifold and let $p \in S$. Suppose we have a global harmonic coordinate system $x = (x^1, x^2, x^3)$ on $(S, h)$ such that $h^{ij}$ is analytic in $x$ and $h_{ij}(p) = \delta_{ij}$. Let $\alpha$ be a smooth positive functions which is analytic in $x$ in a neighborhood of $p$ and with $\alpha(p) = 1$. Let $\widetilde{h} = \alpha^2 h$, and let $\widetilde{x} = (\widetilde{x}^1, \widetilde{x}^2, \widetilde{x}^3)$ be harmonic coordinates on $(S, \widetilde{h})$ such that $\widetilde{x}^a = x^a + O(r(x)^2)$ near $p$, where $r(x) = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. Then $\widetilde{x}^a$ is an analytic function in $x$ and $\widetilde{h}^{ab}$ and $\alpha$ are analytic in $\widetilde{x}$ in a neighborhood of $p$.*

*Proof.* By Lemma 6.1, we have

$$h^{ij}\frac{\partial}{\partial x^i}\left(\alpha\frac{\partial \widetilde{x}^a}{\partial x^j}\right) = 0.$$

Since $\alpha(p) = 1 > 0$ and $h$ is a positive-definite metric, this partial differential equation is elliptic in a neighborhood of $p$. Since the coefficients of the elliptic partial differential equation are analytic, its solutions must also be analytic [80, Theorem 6.6.1]. Therefore, the coordinates $\widetilde{x}^a$ are analytic in $x$. Performing the coordinate transformation shows that $\alpha$ and $\widetilde{h}^{ab}$ are analytic functions in $\widetilde{x}$. □

It is not very unreasonable for such for the coordinates in Lemma 6.2 to exist. Suppose we start with normal coordinates $\big(x^1, x^2, x^3\big)$, i.e., $h_{ij}(p) = \delta_{ij}$ and $\partial_i h_{jk}(p) = 0$, then one can show that there exist coordinates $\big(y^1, y^2, y^3\big)$ centered at $p$ that satisfy $\frac{\partial y^i}{\partial x^j} = \delta^i_j$ and that solve $\Delta_h y^i = 0$ [17, Appendix, Theorem 45]. In that case, we also have $h_{ij}(p)$ in the $y$-coordinates. Moreover, the metric of an Einstein manifold, i.e., a manifold whose Ricci tensor is proportional (by a constant) to the metric, has analytic components in harmonic coordinates [17, Theorem 5.26].

After these lemmas, we move on to proving the equivalence of the multipole moments. We introduce harmonic coordinates $(x^0, x^1, x^2, x^3)$ on $M$ such that $\frac{\partial}{\partial x^0}$ is a stationary vector field. The following lemma shows that these coordinates on $M$ induce harmonic coordinates on the observer space $S$ with metric $h$ given by (2.2).

**Lemma 6.3.** *Let $(M, g)$ be a stationary spacetime, and let $(x^0, x^1, x^2, x^3)$ be global harmonic coordinates on $(M, g)$ such that $\frac{\partial}{\partial x^0}$ is a timelike Killing vector field. Then $(x^1, x^2, x^3)$ can be seen as global harmonic coordinates on the observer space $(S, h)$.*

*Proof.* In the given coordinates, the flow of $\frac{\partial}{\partial x^0}$ is given by $\theta(t,(x^0, x^1, x^2, x^3)) = (x^0 + t, x^1, x^2, x^3)$. In particular, $x^1, x^2, x^3$ are constant along the integral curves of $\frac{\partial}{\partial x^0}$. The fact that the projection $\pi\colon M \to S$ is a surjective smooth submersion implies that we can see $x^1, x^2, x^3$ as smooth functions on $S$. We easily see that $(x^1, x^2, x^3)$ gives a diffeomorphism between $S$ and an open subset of $\mathbb{R}^3$ because $(x^0, x^1, x^2, x^3)$ is a diffeomorphism between $M$ and an open subset of $\mathbb{R}^4$ and $\pi$ is an open map because it is the quotient map of a continuous group action. With the $(x^1, x^2, x^3)$, $\pi$ globally has the form of the local submersion theorem [72, Theorem 4.12]. In these coordinates,

$$h_{ij} = -g_{00}g_{ij} + g_{0i}g_{0j}.$$

Then the inverse metric on $S$ is

$$h^{ij} = \frac{g^{ij}}{-g_{00}},$$

and the determinant of the metric is given by

$$\det h = -(-g_{00})^2 \det g.$$

Since the metric $g$ is independent of $x^0$ and we are working with harmonic coordinates, we have

$$\frac{\partial}{\partial x^i}\left(\sqrt{-\det g}\, g^{i\beta}\right) = \frac{\partial}{\partial x^\alpha}\left(\sqrt{-\det g}\, g^{\alpha\beta}\right) = 0.$$

Therefore,

$$\frac{\partial}{\partial x^i}\left(\sqrt{\det h}\, h^{ij}\right) = \frac{\partial}{\partial x^i}\left((-g_{00})\sqrt{-\det g}\,\frac{g^{ij}}{-g_{00}}\right) = \frac{\partial}{\partial x^i}\left(\sqrt{-\det g}\, g^{ij}\right) = 0,$$

so $(x^1, x^2, x^3)$ is a harmonic coordinate system for $(S, h)$. □

We are finally able to prove the main result.

**Theorem 6.4.** *Let $(M, g)$ be a stationary spacetime that is asymptotically flat according to both Definition 3.2 and Definition 5.12. Shrink $M$ and the corresponding observer space $S$ as explained in the first paragraph above this section. Moreover, assume that the mass monopole moment in the Geroch–Hansen formalism does not vanish. Then the mass and angular momentum multipole moments from the Thorne and Geroch–Hansen formalism are related by* (6.1) *and* (6.2)*, respectively.*

*Proof.* Following Gürsel [46, Section 2.B], we carry out the proof in three steps. First, we express the gravitational potentials in terms of spherical harmonics. The next step is to choose a suitable conformal factor and the last step is to analyse the resulting tensors on $\widetilde{S}$.

**Step 1: expressing the gravitational potentials in terms of spherical harmonics.** Let $\left(x^0, x^1, x^2, x^3\right)$ be global harmonic coordinates for $(M, g)$ such that $\frac{\partial}{\partial x^0}$ is a timelike Killing vector field and it respects the asymptotically flat form of Definition 5.12. In Theorem 5.13, we saw that the metric is of the form (5.12) and we want to use this form to express the metric $h$ on $S$ and the mass and angular momentum potentials in terms of spherical harmonics. For the metric $h$, we have $h_{ij} = -g_{00}g_{ij} + g_{0i}g_{0j}$. A careful but straightforward analysis shows that

$$h_{ij} = \delta_{ij} + \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} S_{l-1}, \tag{6.4}$$

and

$$h^{ij} = \delta^{ij} + \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} S_{l-1},$$

where $S_l$ is an angular function that only contains spherical harmonics of order at most $l$. Since the coordinate vector field $\frac{\partial}{\partial x^0}$ is assumed to be a timelike Killing vector field, we have $\lambda = -g_{00}$ and equation (5.12a) gives

$$\lambda = -g_{00} = 1 - \frac{2\mathcal{M}}{r} + \frac{2\mathcal{M}^2}{r^2} - \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left( \frac{2(2l-1)!!}{l!} \mathcal{M}_{A_l} N_{A_l} + S_{l-1} \right). \tag{6.5}$$

In these coordinates, the twist one-form $\omega = \omega_\alpha dx^\alpha$ of $\xi = \frac{\partial}{\partial x^0}$ reduces to

$$\omega_0 = 0,$$

and

$$\begin{aligned} \omega_i &= \varepsilon_{i\alpha\beta\gamma} \xi^\alpha \nabla^\beta \xi^\gamma = \varepsilon_{i0\beta\gamma} (g^{\beta\alpha} \nabla_\alpha \xi^\gamma) = \varepsilon_{i0\beta\gamma} g^{\beta\alpha} \Gamma^\gamma{}_{\alpha 0} \\ &= -\frac{1}{2} \varepsilon_{0i\beta\gamma} g^{\beta\alpha} g^{\gamma\delta} \left( \frac{\partial g_{\delta 0}}{\partial x^\alpha} + \frac{\partial g_{\alpha\delta}}{\partial x^0} - \frac{\partial g_{\alpha 0}}{\partial x^\delta} \right) = \frac{1}{2} \varepsilon_{0i\beta\gamma} g^{\beta\alpha} g^{\gamma\delta} \left( \frac{\partial g_{\alpha 0}}{\partial x^\delta} - \frac{\partial g_{\delta 0}}{\partial x^\alpha} \right), \end{aligned}$$

where in the last step we used that the metric $g$ is independent of $x^0$. Because $\varepsilon$ is totally antisymmetric, it suffices to sum $\beta$ and $\gamma$ only over the spatial indices and we can also combine the two terms between brackets giving

$$\omega_i = \varepsilon_{0ijk} g^{j\alpha} g^{k\delta} \frac{\partial g_{\alpha 0}}{\partial x^\delta}.$$

Since the metric is independent of $x^0$, we have

$$\omega_i = \varepsilon_{0ijk} g^{j\alpha} g^{km} \frac{\partial g_{\alpha 0}}{\partial x^m} = \varepsilon_{0ijk} g^{jl} g^{km} \frac{\partial g_{0l}}{\partial x^m} + \varepsilon_{0ijk} g^{j0} g^{km} \frac{\partial g_{00}}{\partial x^m} = \varepsilon_{0ijk} g^{jl} g^{km} \left( \frac{\partial g_{0l}}{\partial x^m} - \frac{\partial g_{00}}{\partial x^m} \frac{g_{0l}}{g_{00}} \right).$$

Using (5.12), a careful analysis shows that

$$\begin{aligned} \omega_i &= \sum_{l=1}^{\infty} \left( -\frac{4l(2l-1)!!}{(l+1)!} \varepsilon_{ijk} \varepsilon_{jma_l} \mathcal{J}_{mA_{l-1}} \frac{\partial}{\partial x^k} \left( \frac{1}{r^{l+1}} N_{A_l} \right) + \frac{1}{r^{l+1}} S_{l-1} \right) \\ &= \sum_{l=1}^{\infty} \left( -\frac{4l(2l-1)!!}{(l+1)!} \mathcal{J}_{A_l} \frac{\partial}{\partial x^i} \left( \frac{1}{r^{l+1}} N_{A_l} \right) + \frac{1}{r^{l+2}} S_l \right), \end{aligned} \tag{6.6}$$

where we recall that $S_{l-1}$ changes in the equality, it is a symbol representing certain type of terms: spherical harmonics of order at most $l-1$. Here, we also used that derivatives of spherical harmonics are of the form [77, Appendix A]

$$\frac{\partial}{\partial x^i} \left( \frac{N_{A_l}}{r^{l+1}} \right) \sim \frac{N_{A_l i}}{r^{l+2}} + \frac{S_{l-1}}{r^{l+2}},$$

where $\sim$ means that they are proportional. A careful analysis and this property then also show that $\omega = df$ with

$$f = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left( -\frac{4l(2l-1)!!}{(l+1)!} \mathcal{J}_{A_l} N_{A_l} + S_{l-1} \right). \tag{6.7}$$

Since we know $\lambda$ and $f$, we also know what the potentials $\phi_M$ and $\phi_J$ look like according to Definition 4.1. Thus, by (6.5) and (6.6) we have that

$$\phi_M = \frac{1-\lambda^2-f^2}{4\lambda} = \frac{\mathcal{M}}{r} + \sum_{l=2}^{\infty} \frac{1}{r^{l+1}} \left( \frac{(2l-1)!!}{l!} \mathcal{M}_{A_l} N_{A_l} + S_{l-1} \right), \tag{6.8}$$

and

$$\phi_J = \frac{-f}{2\lambda} = \sum_{l=1}^{\infty} \frac{1}{r^{l+1}} \left( \frac{2l(2l-1)!!}{(l+1)!} \mathcal{J}_{A_l} N_{A_l} + S_{l-1} \right). \tag{6.9}$$

**Step 2: finding a suitable conformal factor.** Eventually, we want to get a conformal factor of the form $\Omega = \frac{1}{r} + \sum_{l=2}^{\infty} \frac{S_{l-1}}{r^{l+1}}$ because it would not change the highest order terms in the potentials when moving from $\phi_A$ to $\widetilde{\phi}_A$. First, consider the smooth positive function

$$\Omega_{BS} = \frac{1}{2B^2} \left( \sqrt{1 + 4\phi_M^2 + 4\phi_J^2} - 1 \right).$$

Here, $B$ is a constant such that $\widetilde{D}_i \widetilde{D}_j \Omega_{BS}|_{i^0} = 2\widetilde{h}_{ij}|_{i^0}$, for which we find

$$B^2 = \left( \widetilde{\phi}_M^{BS}(i^0) \right)^2 + \left( \widetilde{\phi}_J^{BS}(i^0) \right)^2.$$

By assumption, the mass monopole moment does not vanish, so $\widetilde{\phi}_M^{BS}(i^0) \neq 0$ and also $B \neq 0$. The superscript $BS$ means that $\Omega_{BS}$ acts as a conformal factor, so $\widetilde{h}^{BS} = \Omega_{BS}^2 h$ and $\widetilde{\phi}_A^{BS} = \Omega_{BS}^{-\frac{1}{2}} \phi_A$. We write $BS$ because $\Omega_{BS}$ is due to Beig and Simon [13]. They have proven that we can always take $\Omega_{BS}$ as a conformal factor and it satisfies a remarkable property: with harmonic coordinates on a neighborhood of $i^0$, the functions $\widetilde{h}_{ij}^{BS}$, $\Omega_{BS}$ and $\widetilde{\phi}_A^{BS}$ are analytic [13, Theorem 1]. Alternatively, we can also consider

$$\Omega_G = \frac{\phi_M^2}{M^2} = \alpha \Omega_{BS},$$

with $M = \widetilde{\phi}_M^{BS}(i^0)$, which we assumed to be nonzero, and where

$$\alpha = \frac{2B^2}{M^2} \frac{\phi_M^2}{\sqrt{1 + 4\phi_M^2 + 4\phi_J^2} - 1} = \frac{\left( \widetilde{\phi}_M^{BS} \right)^2}{M^2}.$$

Hence, $\alpha$ is analytic in our chosen coordinates and $\alpha(i^0) = 1$. This turns $\Omega_G$ also into a suitable conformal factor. Lemma 6.2 tells us that we can also take harmonic coordinates around $i^0$ for the metric induced by $\Omega_G$, and the conformal metric, conformal factor and conformal potential fields are also analytic in these harmonic coordinates. For the power expansion, a tedious calculation yields [46, Eq. (43)]

$$\Omega_G = \frac{1}{r^2}\left(1+\sum_{l=2}^{\infty}\frac{1}{r^l}(\mathcal{I}_{A_l}N_{A_l}+S_{l-1})\right), \tag{6.10}$$

where

$$\mathcal{I}_{A_l} = \frac{4(2l-1)!!}{l!}\frac{\mathcal{J}_{A_l}}{2M}+\sum_{k=2}^{l-2}\frac{2(2k-1)!!}{k!}\frac{2(2(l-k)-1)!!}{(l-k)!}\frac{\mathcal{J}_{A_k}}{2M}\frac{\mathcal{J}_{a_{k+1}\dots a_l}}{2M}.$$

Recall that in step 1 we take harmonic coordinates $(x^0,x^1,x^2,x^3)$ for $M$ such that $\frac{\partial}{\partial x^0}$ is a stationary vector field. By Lemma 6.3, $(x^1,x^2,x^3)$ give smooth coordinates on $S$. Let $(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)$ be global harmonic coordinates on $(\widetilde{S},\widetilde{h})$ centered at $i^0$ (note that we can shrink $S$ if necessary as long as $\widetilde{S}$ contains $i^0$ as an interior point). Then by Lemma 6.1 we have

$$h^{ij}\frac{\partial}{\partial x^i}\left(\Omega_G\frac{\partial\widetilde{x}^a}{\partial x^j}\right)=0, \tag{6.11}$$

on $S$. Conversely, Lemma 6.1 also tells us that coordinates $(\widetilde{x}^1,\widetilde{x}^2,\widetilde{x}^3)$ centered at $i^0$ solving (6.11) are necessarily harmonic with respect to $\widetilde{h}$ except at $i^0$. So, they solve $\widetilde{\Delta}_{\widetilde{h}}\widetilde{x}^i=0$ except possibly at $i^0$. But then it must hold at $i^0$ by continuity. Equation (6.11) admits a solution of the form [46, Eq. (45)]

$$\widetilde{x}^a = \frac{x^a}{r^2}\left(1+\sum_{l=2}^{\infty}\frac{1}{r^l}\mathcal{A}_{A_l}N_{A_l}\right)+\sum_{l=2}^{\infty}\frac{1}{r^{l+1}}\big(\mathcal{B}_{aA_{l-1}}N_{A_{l-1}}+S_{l-1}\big), \tag{6.12}$$

where $\mathcal{A}_{A_l}$ and $\mathcal{B}_{A_l}$ are constants which can be written in terms of Thorne's multipole moments. By adapting the conformal factor and the coordinates, we can simplify it further. We consider the conformal factor $\Omega' = \beta\Omega_G$ with

$$\beta = 1+\sum_{l=2}^{\infty}\mathcal{C}_{A_l}\widetilde{x}^{a_1}\cdots\widetilde{x}^{a_l}, \tag{6.13}$$

where $\mathcal{C}_{A_l}$ are arbitrary constants and we consider coordinates

$$x'^a = \widetilde{x}^a\left(1+\sum_{l=2}^{\infty}\mathcal{D}_{A_l}\widetilde{x}^{a_1}\cdots\widetilde{x}^{a_l}\right)+\sum_{l=2}^{\infty}\mathcal{E}_{aA_{l-1}}\widetilde{x}^{a_1}\cdots\widetilde{x}^{a_{l-1}}\widetilde{r}^2, \tag{6.14}$$

for some arbitrary constants $\mathcal{D}_{A_l}$ and $\mathcal{E}_{A_l}$. We want to choose the constants such that

$$x'^a = \frac{x^a}{r^2}+\frac{1}{r}\sum_{l=2}^{\infty}\frac{S_{l-1}}{r^l}, \tag{6.15}$$

and

$$\Omega' = \omega\Omega = \frac{1}{r}\left(1+\sum_{l=2}^{\infty}\frac{S_{l-1}}{r^l}\right). \tag{6.16}$$

Equation (6.15) constitutes relations between $\mathcal{A}_{A_l}$, $\mathcal{B}_{A_l}$, $\mathcal{D}_{A_l}$ and $\mathcal{E}_{A_l}$ via (6.12) and (6.14) and (6.16) constitutes relations between $\mathcal{A}_{A_l}$, $\mathcal{B}_{A_l}$, $\mathcal{C}_{A_l}$, $\mathcal{I}_{A_l}$ via (6.10), (6.12) and (6.13). We consider $\mathcal{A}_{A_l}$, $\mathcal{B}_{A_l}$, $\mathcal{I}_{A_l}$ as given and it is possible to solve these equations for $\mathcal{C}_{A_l}$, $\mathcal{D}_{A_l}$ and $\mathcal{E}_{A_l}$ [46].

**Step 3: analysing the tensors at $i^0$.** Using (6.4), (6.15) and (6.16), the conformal metric with conformal factor $\Omega'$ is

$$\widetilde{h}'_{a'b'} = \omega^2 \widetilde{h}_{\widetilde{c}\widetilde{d}} \frac{\partial \widetilde{x}^{\widetilde{c}}}{\partial x'^{a'}} \frac{\partial \widetilde{x}^{\widetilde{d}}}{\partial x'^{b'}} = \delta_{a'b'} + \sum_{l=2}^{\infty} r'^l S_{l-1}.$$

The gravitational field potentials become, using (6.8) and (6.16),

$$\widetilde{\Phi}_{,M} = M + \sum_{l=2}^{\infty} \left( \frac{(2l-1)!!}{l!} \mathcal{M}_{A'_l} x'^{a'_1} \cdots x'^{a'_l} + r'^l S_{l-1} \right),$$

and, using (6.9) and (6.16),

$$\widetilde{\Phi}_J = \sum_{l=1}^{\infty} \left( \frac{2l(2l-1)!!}{(l+1)!} \mathcal{J}_{A'_l} x'^{a'_1} \cdots x'^{a'_l} + r'^l S_{l-1} \right).$$

Because the spherical harmonics in the $r'^l S_{l-1}$-part of the metric are only of order at most $l-1$, we see that the Christoffel symbols and the Ricci tensor terms vanish in the limit when evaluating at $i^0$, which is found by setting $r' = 0$. Therefore, for the Geroch–Hansen formalism determined by (4.17) we see that the only term of interest are the derivative terms. Taking the relevant derivatives of the $x'^{a'_1} \cdots x'^{a'_l}$-terms in the conformal potentials returns indeed equations (6.1) and (6.2). □

## 6.2 Additional properties of multipole moments in the literature

We end this chapter by mentioning three results. One of the advantages of Theorem 6.4 is that we can choose to work with either the Geroch-Hansen or the Thorne formalism (assuming the conditions of both formalisms are satisfied).

The first result is about static spacetimes. If $(M, g)$ is a static spacetime, the twist covector field vanishes. Hence, the angular momentum potential also vanishes and we see the angular momentum multipole moments all vanish. Equivalently, we can take coordinates such that the metric satisfies $g_{0i} = 0$, from which we conclude the angular momentum multipole moments all vanish by Thorne's formalism. The following result tells us that the converse is also true.

**Theorem 6.5.** *A stationary, asymptotically flat, vacuum spacetime is static if and only if all angular momentum multipole moments vanish.*

*Proof.* If the spacetime is static, the proof is already given in the paragraph above. The converse is proven by Xanthopoulos [116] in 1979 in the Geroch-Hansen formalism. It would be way more difficult to prove it in the Thorne formalism [46]. □

The second result is about axisymmetric spacetimes. As we saw in Section 4.3, the multipole moments for axisymmetric spacetimes are also axisymmetric. The converse is also true.

**Theorem 6.6.** *A stationary, asymptotically flat, vacuum spacetime is axisymmetric if and only if all multipole moments are axisymmetric.*

*Proof.* If the spacetime is axisymmetric, it follows that the multipole moments are axisymmetric as in Section 4.3. Similarly, if the metric components are independent of some angular coordinate $\varphi$, we see that only the spherical harmonics with $m = 0$ can appear, so the multipole moments are also axisymmetric in the Thorne formalism. The converse follows by carrying out the derivation of the Thorne formalism [46]. □

The last result is the most important one. Where the two results above characterise spacetime properties using multipole moments, this result characterises the spacetime itself. Two vacuum solutions of the Einstein equations with the same multipole moments must look the same near infinity. It means that the spacetime, except for a bounded region, is characterised by its multipole moments.

**Theorem 6.7.** *Two stationary, asymptotically flat, vacuum spacetimes with the same multipole moments are isometric in a neighborhood of $i^0$.*

*Proof.* This is independently proven by Beig and Simon [12, 13] and [67], both in 1981, in the Geroch-Hansen formalism. It can also be proven by carrying out the derivation of the Thorne formalism [46]. □

# Part III

# Multipole Moments in Spacetimes with Matter

# Chapter 7

# Geometric multipole moments in electrovacuum

In Part II, we studied multipole moments in vacuum solutions of Einstein's equations. We want to generalise these multipole moments to non-vacuum solutions. In this chapter, we want to consider electrovacuum solutions. Electrovacuum solutions are spacetimes solving the Einstein–Maxwell equations without sources. In 1984, ten years after Hansen's definition of multipole moments [48], Simon generalised the Geroch–Hansen multipole moments to electrovacuum [102]. The approach by Simon is discussed in Section 7.1. In Section 7.2, we study how the multipole moments simplify in axisymmetric spacetimes like we have done for vacuum in Section 4.3.

## 7.1 Multipole moments

Recall from Chapter 2 that a stationary spacetime $(M, g)$ has a complete timelike Killing vector field $\xi$ and comes naturally with the observer space $(S, h)$, which is a three-dimensional Riemannian manifold. We assume it is asymptotically flat in the sense of Geroch, according to Definition 3.2. Like we did in vacuum, we only care about the situation at infinity, so we can restrict the observer space such that $S$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B}^3$. Correspondingly, the restricted spacetime is diffeomorphic to $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \overline{B}^3\right)$. Therefore, we can assume that $H^1_{\mathrm{dR}}(S) = H^1_{\mathrm{dR}}(M) = 0$ and $H^2_{\mathrm{dR}}(S) = H^2_{\mathrm{dR}}(M) = \mathbb{R}$ because $S$ and $M$ are homotopy equivalent to $\mathbb{S}^2$, like we argued in the second paragraph of Chapter 4.

In electrovacuum, we do not only want that the spacetime is stationary but also that the electromagnetic field to be stationary. This leads to the following definitions.

**Definition 7.1.** An *electromagnetic field tensor* without sources is a closed 2-form $F$ such that $*d*F = 0$. It is called *exact* if $F$ is exact, in which case $F = dA$ for an *electromagnetic potential* $A$.

**Definition 7.2.** The electromagnetic field $F$ in a stationary spacetime $(M, g)$ with a stationary vector field $\xi$ is *stationary* if $\mathcal{L}_\xi F = 0$. An exact electromagnetic field $F = dA$ is called *stationary* if $\mathcal{L}_\xi A = 0$.

Note that we assumed that $H^1_{\mathrm{dR}}(M) = 0$ and $H^2_{\mathrm{dR}}(M) = \mathbb{R}$. In particular, there are closed 2-forms which are not exact, so an electromagnetic field tensor does not need to be exact. Suppose $F = dA = dA'$ for $A, A' \in \Omega^1(M)$. Then $A - A'$ is a closed one-form and $H^1_{\mathrm{dR}}(M) = 0$ implies that $A - A'$ is an exact one-form on $M$. So the potential for an exact electromagnetic field can only differ by differentials of functions.

Since an electromagnetic field does not need to be exact, it is an extra assumption that may be unwanted. In the research papers [38, 55, 77, 102], exactness is assumed, but we take a slightly different approach. Eventually, we are only interested in the scalar potentials, and they can also be defined as potentials for closed one-forms. Since $H^1_{\mathrm{dR}}(M) = 0$ for the restricted spacetime as above, it makes more sense to assume exactness of closed one-forms rather than exactness of closed two-forms.

**Proposition 7.3.** *Let $(M, g)$ be a stationary spacetime with $H^1_{\mathrm{dR}}(M) = 0$ and let $F$ be a stationary electromagnetic field without sources on $(M, g)$. Then there exist functions $\varphi_E, \varphi_B \in C^\infty(M)$ such that*

$$d\varphi_E = i_\xi F, \qquad d\varphi_B = i_\xi * F. \tag{7.1}$$

*Proof.* From Cartan's magic formula we get

$$di_\xi F = \mathcal{L}_\xi F - i_\xi dF = 0,$$

because $F$ is stationary and closed. Therefore, $i_\xi F$ is a closed one-form on $M$. Since $H^1_{\mathrm{dR}}(M) = 0$, the one-form $i_\xi F$ is exact, so there exists a smooth scalar field $\varphi_E \in C^\infty(M)$ such that $i_\xi F = d\varphi_E$. Similarly, we also have

$$di_\xi * F = \mathcal{L}_\xi * F - i_\xi d * F = *\mathcal{L}_\xi F = 0.$$

Here, we note that $*d * F = 0$ implies that $d * F = 0$ and $\mathcal{L}_\xi$ commutes with $*$ because $\xi$ is a Killing vector field. So, there also exists a smooth scalar field $\varphi_B \in C^\infty(M)$ such that $i_\xi * F = d\varphi_B$. □

We can see the scalar fields $\varphi_E$ and $\varphi_B$ satisfying (7.1) as electric and magnetic scalar potentials, respectively. Moreover, observe that

$$\mathcal{L}_\xi \varphi_E = i_\xi d\varphi_E = i_\xi i_\xi F = 0,$$

and

$$\mathcal{L}_\xi \varphi_B = i_\xi d\varphi_B = i_\xi i_\xi * F = 0,$$

so $\varphi_E$ and $\varphi_B$ are scalar fields that live on the observer space $S$ by Proposition 2.6. They are only defined up to a constant by (7.1), but we will see at the end of this section that there is no gauge freedom under an extra assumption.

For an exact electromagnetic field, we usually decompose the four-potential into the electric scalar potential and the magnetic vector potential. Suppose $F$ is an exact, stationary electromagnetic field that is also co-exact such that $F = dA$ and $*F = d\widetilde{A}$ for some $A, \widetilde{A} \in \Omega^1(M)$ with $\mathcal{L}_\xi A = \mathcal{L}_\xi \widetilde{A} = 0$. Then,

$$d(A(\xi)) = di_\xi A = \mathcal{L}_\xi A - i_\xi dA = -i_\xi dA = -i_\xi F,$$

implying that $\varphi_E$ equals $-A(\xi)$ up to a constant. Similarly,

$$d\Big(\widetilde{A}(\xi)\Big) = di_\xi \widetilde{A} = \mathcal{L}_\xi \widetilde{A} - i_\xi d\widetilde{A} = -i_\xi * F,$$

implying that $\varphi_B$ equals $-\widetilde{A}(\xi)$ up to a constant. With coordinates such that $\xi = \frac{\partial}{\partial t}$, $-A_t$ is interpreted as the electric scalar potential and $-\widetilde{A}_t$ is interpreted as the magnetic scalar potential. Therefore, we can interpret $\varphi_E$ and $\varphi_B$ in the same way.

The scalar fields $\varphi_E$ and $\varphi_B$ play very important roles because they contain a lot of information about the electromagnetic field. To define multipole moments in the Geroch–Hansen formalism, we need to find potentials (scalar fields) that contain a lot of information. The next result shows that the electromagnetic field is completely determined by $\varphi_E$ and $\varphi_B$!

**Proposition 7.4.** *Let $(M, g)$ be a stationary spacetime with $H^1_{\mathrm{dR}}(M) = 0$ and let $F$ be a stationary electromagnetic field without sources on $(M, g)$. Let $\varphi_E, \varphi_B \in C^\infty(M)$ be scalar fields satisfying* (7.1)*, then*

$$F = -\lambda^{-1}\xi^\flat \wedge d\varphi_E + \lambda^{-1} * \Big(\xi^\flat \wedge d\varphi_B\Big). \tag{7.2}$$

*Proof.* For the magnetic scalar potential, we have

$$d\varphi_B = i_\xi * F = *\Big(\xi^\flat \wedge F\Big),$$

so $\xi^\flat \wedge F = *d\varphi_B$. Then,

$$-\lambda F - \xi^\flat \wedge d\varphi_E = -\lambda F - \xi^\flat \wedge i_\xi F = i_\xi\Big(\xi^\flat \wedge F\Big) = i_\xi * d\varphi_B = - * \Big(\xi^\flat \wedge d\varphi_B\Big).$$

Rewriting this equation gives (7.2). □

In coordinates, (7.2) reads

$$F_{\mu\nu} = -\lambda^{-1}(\xi_\mu \partial_\nu \varphi_E - \xi_\nu \partial_\mu \varphi_E) + \lambda^{-1}\varepsilon_{\mu\nu\rho\sigma}\xi^\rho g^{\sigma\tau}\partial_\tau \varphi_B. \tag{7.3}$$

In the construction of the Geroch–Hansen multipole moments, the twist one-form defined by (2.5) plays an important role. The exterior derivative of the twist one-form can be expressed in terms of the Ricci tensor by Theorem 2.12. Therefore, we want to calculate the Ricci tensor in terms of $\varphi_E$ and $\varphi_B$. The Einstein–Maxwell equations imply that the scalar curvature vanishes and for the Ricci tensor we have

$$R_{\mu\nu} = 2\bigg(F_{\mu\rho}F_\nu{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\bigg). \tag{7.4}$$

From (7.2) or (7.3), one can find

$$\begin{aligned} F_{\mu\rho}F_\nu{}^\rho = {} & \lambda^{-2}\xi_\mu\xi_\nu\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big) + 2\lambda^{-2}\xi_{(\mu}\varepsilon_{\nu)}{}^{\rho\sigma\tau}\xi_\rho\partial_\sigma\varphi_E\partial_\tau\varphi_B \\ & - \lambda^{-1}(\partial_\mu\varphi_E\partial_\nu\varphi_E + \partial_\mu\varphi_B\partial_\nu\varphi_B) + \lambda^{-1}g_{\mu\nu}|d\varphi_B|_g^2, \end{aligned}$$

and

$$F_{\rho\sigma}F^{\rho\sigma} = -2\lambda^{-1}|d\varphi_E|_g^2 + 2\lambda^{-1}|d\varphi_B|_g^2.$$

Therefore, the Ricci tensor (7.4) is

$$R_{\mu\nu} = 2\lambda^{-2}\xi_\mu\xi_\nu\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big) + 4\lambda^{-2}\xi_{(\mu}\varepsilon_{\nu)}{}^{\rho\sigma\tau}\xi_\rho\partial_\sigma\varphi_E\partial_\tau\varphi_B \\ - 2\lambda^{-1}(\partial_\mu\varphi_E\partial_\nu\varphi_E + \partial_\mu\varphi_B\partial_\nu\varphi_B) + \lambda^{-1}g_{\mu\nu}\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big),$$

In the mathematical, global notation it reads

$$Rc = 2\lambda^{-2}\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big)\xi^\flat\otimes\xi^\flat - 2\lambda^{-2}\xi^\flat\otimes *\Big(\xi^\flat\wedge d\varphi_E\wedge d\varphi_B\Big) - 2\lambda^{-2} * \Big(\xi^\flat\wedge d\varphi_E\wedge d\varphi_B\Big)\otimes\xi^\flat \\ - 2\lambda^{-1}(d\varphi_E\otimes d\varphi_E + d\varphi_B\otimes d\varphi_B) + \lambda^{-1}\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big)g.$$

Therefore,

$$Rc(\xi,\cdot) = -\lambda^{-1}\Big(|d\varphi_E|_g^2 + |d\varphi_B|_g^2\Big)\xi^\flat + 2\lambda^{-1}\Big(*\Big(\xi^\flat\wedge d\varphi_E\wedge d\varphi_B\Big)\Big).$$

By Theorem 2.12, the exterior derivative of the twist one-form becomes

$$d\omega = -2*\Big(\xi^\flat\wedge Rc(\xi,\cdot)\Big) = 4\lambda^{-1}i_\xi**\Big(\xi^\flat\wedge d\varphi_E\wedge d\varphi_B\Big) = 4\lambda^{-1}i_\xi\Big(\xi^\flat\wedge d\varphi_E\wedge d\varphi_B\Big) = -4d\varphi_E\wedge d\varphi_B.$$

Let $\omega^I \in \Omega^1(M)$ be given by

$$\omega^I = 2(\varphi_B d\varphi_E - \varphi_E d\varphi_B), \tag{7.5}$$

then

$$d\omega^I = 4d\varphi_B\wedge d\varphi_E.$$

So, $\omega + \omega^I$ is a closed one-form on $M$. It will replace the role of the twist one-form in vacuum. Still, we have $i_\xi\omega = 0$ and $\mathcal{L}_\xi\omega = 0$. Moreover, $i_\xi\omega^I = 0$ and $\mathcal{L}_\xi\omega^I = 0$ because $\mathcal{L}_\xi\varphi_E = \mathcal{L}_\xi\varphi_B = 0$. Therefore, by Proposition 2.6, there is a covector field $\omega'$ on the observer space $S$ such that $\pi^*\omega' = \omega + \omega^I$. We see that $\omega'$ must also be closed because the pullback by a surjective submersion is injective. Under the reasonable assumption that $H^1_{\mathrm{dR}}(S) = 0$, this gives a smooth scalar field $f$ on $S$ such that $\omega' = df$.

On $S$, we have the scalar fields $\lambda$, $f$, $\varphi_E$ and $\varphi_B$. In Section 4.1 we define the Ernst potential $\mathcal{E}$, but we need to adapt it for Einstein-Maxwell solutions. Define the complex scalar fields

$$\varphi_B = \varphi_E + i\varphi_B,$$

and

$$\mathcal{E} = \lambda + if - \varphi_E^2 - \varphi_B^2.$$

Then, define $\xi, q \in C^\infty(S)$ by

$$\xi = \frac{1-\mathcal{E}}{1+\mathcal{E}}, \qquad q = \frac{2\varphi_B}{1+\mathcal{E}}. \tag{7.6}$$

Decomposing the real and imaginary parts as $\xi = \phi_M + i\phi_J$ and $q = \phi_E + i\phi_B$ gives[12]

$$\phi_M = \frac{1 - f^2 - \big(\lambda - \varphi_E^2 - \varphi_B^2\big)^2}{\big(1 + \lambda - \varphi_E^2 - \varphi_B^2\big)^2 + f^2}, \tag{7.7a}$$

[12] Excuse me for the bad notation with $\varphi_{B,E}$ and $\phi_{B,E}$.

$$\phi_J = \frac{-2f}{\left(1+\lambda-\varphi_E^2-\varphi_B^2\right)^2+f^2}, \tag{7.7b}$$

$$\phi_E = \frac{2\varphi_E\left(1+\lambda-\varphi_E^2-\varphi_B^2\right)+2\varphi_B f}{\left(1+\lambda-\varphi_E^2-\varphi_B^2\right)^2+f^2}, \tag{7.7c}$$

$$\phi_B = \frac{2\varphi_B\left(1+\lambda-\varphi_E^2-\varphi_B^2\right)-2\varphi_E f}{\left(1+\lambda-\varphi_E^2-\varphi_B^2\right)^2+f^2}. \tag{7.7d}$$

**Definition 7.5.** For a stationary solution of the Einstein-Maxwell equations, the four scalar fields $\phi_M$, $\phi_J$, $\phi_E$, and $\phi_B$ given in equation (7.7) are the *mass*, *angular momentum*, *electric*, and *magnetic potential*, respectively.

With these potentials, we are in business. The construction around multipole moments in Section 4.2 and in Definition 4.9 specifically, does not depend on the mass and angular momentum potentials in vacuum. We can readily apply the same recurrence relation.

**Definition 7.6.** Let $(S,h)$ be an asymptotically flat Riemannian manifold whose one-point extension is $(\widetilde{S},\widetilde{\phi})$ and such that we have the mass, angular momentum, electric, and magnetic potentials $\phi_M$, $\phi_J$, $\phi_E$, and $\phi_B$, respectively. Suppose $\widetilde{\phi}_A = \Omega^{-\frac{1}{2}}\phi_A$ extends to a smooth function on $\widetilde{S}$ for $A = M, J, E, B$. Then the *mass*, *angular momentum*, *electric*, and *magnetic $2^k$-pole moment* are the $2^k$-pole moments of $\phi_M$, $\phi_J$, $\phi_E$, and $\phi_B$, respectively, which are defined in Definition 4.5.

For solutions of the Einstein-Maxwell equations, we certainly want to recover the multipole moments in vacuum once the electromagnetic field is turned off. If $F = 0$, we can take $\varphi_E = 0$ and $\varphi_B = 0$, so that the electric and magnetic potentials vanish. Moreover, the mass and angular momentum potentials are just the gravitational potentials in vacuum from Definition 4.1. Hence, we find the same multipole moments.

Since we use exactly the same recurrence relation to define multipole moments, Proposition 4.10 and Corollary 4.11 are still valid. So, the electromagnetic multipole moments also obey the same conformal transformation laws.

We end this section with briefly looking at the smoothness condition of the $\widetilde{\phi}_A$ in Definition 7.6. It would be interesting to find out whether there are results like Lemma 4.4 that significantly weaken the smoothness condition by using elliptic regularity. However, we do not delve into that issue here. We do want to observe some convergence properties. We must have $\phi_A \to 0$ when $x \to i^0$ for $A = M, J, E, B$. Like in vacuum, which we discussed at the end of Section 4.1, the expressions for $\phi_M$ and $\phi_J$ imply that $\lambda - \varphi_E^2 - \varphi_B^2 \to 1$ and $f \to 0$. Then the equations for $\phi_E$ and $\phi_B$ imply that $\varphi_E \to 0$ and $\varphi_B \to 0$. But then we also have $\lambda \to 1$ again, so we have exactly the same convergence properties as expected.

## 7.2 Axisymmetric spacetimes and the Kerr–Newman solution

Like in the Geroch–Hansen formalism in vacuum, it is difficult to calculate the multipole moments. However, if the spacetime is also axisymmetric, we can simplify the calculation of the multipole moments greatly as we saw in Section 4.3. In this section, we basically want to redo everything in Section 4.3 but now in electrovacuum. The goal is to calculate the multipole moments for the Kerr–Newman spacetime. This section is split into three parts. First, we discuss the assumptions we need and see how the multipole moments simplify. After that, we discuss the generalised Fodor–Hoenselaers–Perjés and Bäckdahl–Herberthson algorithms. Finally, we calculate the multipole moments of the Kerr–Newman solution.

### Multipole moments in an axisymmetric electrovacuum

Recall from Definition 4.12 that a stationary spacetime is axisymmetric if it admits a spacelike Killing vector field $\psi$ with closed flow lines and such that it commutes with the stationary vector field. It is not sufficient to require only that the spacetime is axisymmetric. We also want the electromagnetic field to be axisymmetric.

**Definition 7.7.** Let $(M, g)$ be a stationary axisymmetric spacetimes with axisymmetric vector field $\psi$ and let $F$ be a stationary electromagnetic field without sources on $(M, g)$. Then $F$ is called *axisymmetric* if $\mathcal{L}_\psi F = 0$.

So, the electromagnetic field is not only time-invariant but also rotation-invariant. In that case, it is shown by Carter [26] that $F(\xi, \psi) = *F(\xi, \psi) = 0$ under reasonable assumptions. For example, it is sufficient to assume that the spacetime is connected (which we assume anyway) and the vector field $\psi$ vanishes at some point (which usually happens on the rotation axis in asymptotically flat spacetimes). If $F(\xi, \psi) = *F(\xi, \psi) = 0$, then the Weyl-Lewis-Papapetrou coordinates can also be used in electrovacuum [106, Chapter 19]. So, the metric is of the form

$$g = -\lambda(dt - wd\varphi)^2 + \lambda(\rho^2 d\varphi^2 + e^{2\gamma}(d\rho^2 + dz^2)), \tag{7.8}$$

where $\lambda$, $w$ and $\gamma$ are functions that only depend on $\rho$ and $z$. In these coordinates, $F(\xi, \psi) = *F(\xi, \psi) = 0$ can be formulated as $F_{t\varphi} = 0$ and $(*F)_{t\varphi} = 0$. But with the metric of the form given by (7.8), $(*F)_{t\varphi} = 0$ implies that $F_{\rho z} = 0$. Therefore, the only nonzero components of $F$ are $F_{t\rho}$, $F_{tz}$, $F_{\rho\varphi}$ and $F_{z\varphi}$ (and their antisymmetric counterparts). Moreover, these component functions can only depend on $\rho$ and $z$ because $\mathcal{L}_\xi F = \mathcal{L}_\psi F = 0$. Since $F(\xi, \psi) = *F(\xi, \psi) = 0$, we get for the electromagnetic scalar potentials $\varphi_E$ and $\varphi_B$ that

$$\mathcal{L}_\psi \varphi_E = i_\psi d\varphi_E = i_\psi i_\xi F = F(\xi, \psi) = 0,$$

and

$$\mathcal{L}_\psi \varphi_B = i_\psi d\varphi_B = i_\psi i_\xi * F = *F(\xi, \psi) = 0.$$

So, $\varphi_E$ and $\varphi_E$ are also not only invariant under time translations but also under rotations. From (7.5) we also see that $i_\psi \omega^I = 0$. In vacuum, we concluded that $i_\psi \omega = 0$ in (4.35) based on the orthogonal form of the metric. But those conditions also hold in electrovacuum, so we also have $i_\psi \omega = 0$ here. Therefore,

$$\mathcal{L}_\psi f = i_\psi df = i_\psi \left(\omega + \omega^I\right) = 0.$$

Hence, also $f$ is rotation-invariant. For $\lambda$, we have exactly the same reasoning as in vacuum to conclude it is rotation-invariant. Therefore, the potentials determined by (7.7) are also rotation-invariant and we can use the reasoning in Section 4.3 in verbatim to conclude that all multipole moments must be multiples of $(d\widetilde{z} \otimes \cdots \otimes d\widetilde{z})^{STF}$ in the notation from Section 4.3.

### First algorithm to find multipole moments

In this part, we want to discuss the Fodor–Hoenselaers–Perjés algorithm to electrovacuum. One year after it was found in vacuum, this generalised algorithm was published by Hoenselaers and Perjés [54] in 1990. However, there were some mistakes. In 2004, some corrections were made by Sotiriou and Apostolatos [105], but there was still one error left. Luckily, the issue has been solved by Fodor, Costa Filho and Hartmann [38] in 2021. Since the derivation is exactly the same as in vacuum but the expressions are even worse and the calculations are even more tedious, we restrict ourselves to giving the results. The details can be found in the three papers cited above.

**Theorem 7.8.** *Suppose we have a stationary axisymmetric, asymptotically flat electrovacuum solution of the Einstein equations and the potentials* $\widetilde{\xi} = \Omega^{-\frac{1}{2}}\xi$ *and* $\widetilde{q} = \Omega^{-\frac{1}{2}}q$ *determined by* (7.6) *are analytic around* $i^0$*, then the first* $m+1$ *multipole moments can be computed using the following algorithm:*

1. *Find the coefficients* $a_{0j}$ *and* $b_{0j}$ *for* $j \leq m$ *by* $\widetilde{\xi}\Big|_{\widetilde{\rho}=0} = \sum_{j=0}^{\infty} a_{0j}\widetilde{z}^j$ *and* $\widetilde{q}|_{\widetilde{\rho}=0} = \sum_{j=0}^{\infty} b_{0j}\widetilde{z}^j$*;*
2. *Determine* $a_{ij}$ *and* $b_{ij}$ *for* $i+j \leq m$ *using the recursion relations given by [38, Section IV.B]*

$$\begin{aligned}(r+s)^2 a_{r+2,s} = &-(s+2)(s+1)a_{r,s+2}\\ &+\sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)a_{pq}\\ &\quad\times(p^2+q^2-2p-3q-2k-2l-2pk-2ql-2)\\ &+\sum_{p=-2}^{r}\sum_{q=2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)a_{p+2,q-2}(p+2)(p+2-2k)\\ &+\sum_{p=2}^{r}\sum_{q=-2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)a_{p-2,q+2}(q+2)(q+1-2l),\end{aligned}$$

*and*

$$\begin{aligned}(r+s)^2 b_{r+2,s} = &-(s+2)(s+1)b_{r,s+2}\\ &+\sum_{p=0}^{r}\sum_{q=0}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)b_{pq}\\ &\quad\times(p^2+q^2-2p-3q-2k-2l-2pk-2ql-2)\\ &+\sum_{p=-2}^{r}\sum_{q=2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)b_{p+2,q-2}(p+2)(p+2-2k)\\ &+\sum_{p=2}^{r}\sum_{q=-2}^{s}\sum_{k=0}^{r-p}\sum_{l=0}^{s-q}\left(a_{kl}\overline{a_{mn}} - b_{kl}\overline{b_{mn}}\right)b_{p-2,q+2}(q+2)(q+1-2l),\end{aligned}$$

   *where* $m = r-p-k$ *and* $n = s-q-l$*. Like in vacuum, we assume that* $a_{ij}$ *and* $b_{ij}$ *vanish whenever* $i$ *is odd;*

3. *Calculate the components of the Ricci tensor* $\widetilde{R}_{ij}$ *in terms of* $a_{ij}$ *and* $b_{ij}$ *using*

$$\Theta^2 \widetilde{R}_{ij} = 2\,\mathrm{Re}\Big(\widetilde{D}_i \widetilde{xi} \widetilde{D}_j \overline{\widetilde{\xi}} - \widetilde{D}_i \widetilde{q} \widetilde{D}_j \overline{\widetilde{q}} + \widetilde{s}_i \overline{\widetilde{s}}_j\Big),$$

*where*

$$\Theta = \widetilde{r}^2 \widetilde{\xi} \overline{\widetilde{\xi}} - \widetilde{r}^2 \widetilde{q} \overline{\widetilde{q}} - 1,$$

*and*

$$\widetilde{s}_i = \widetilde{r}\Big(\widetilde{\xi} \widetilde{D}_i \widetilde{q} - \widetilde{q} \widetilde{D}_i \widetilde{\xi}\Big),$$

*with* $\widetilde{r} = \sqrt{\widetilde{\rho}^2 + \widetilde{z}^2} = \frac{1}{\sqrt{\rho^2 + z^2}}$ *because* $\widetilde{\rho} = \frac{\rho}{\rho^2 + z^2}$ *and* $\widetilde{z} = \frac{z}{\rho^2 + z^2}$. *The derivatives of* $\gamma$ *can then still be determined using* (4.42)*;*

4. *Compute* $S^n_a$ *for* $n \leq m$ *and* $a \leq m - n$ *using* (4.47) *and* (4.48)*, once with* $S^0_0 = \widetilde{\xi}$ *and once with* $S^0_0 = \widetilde{q}$. *We only need to know* $S^n_a$ *up to degree* $\widetilde{\rho}^k \widetilde{z}^l$ *with* $k + l \leq m - n$*;*

5. *Evaluating* (4.49) *for* $n = 0, 1, \ldots, m$ *and using* (4.37) *to find the multipole moments.*

*Proof.* See [54] with the necessary corrections in [38, 105] □

A (correct) explicit expression for the first seven multipole moments in terms of $a_{0j}$ and $b_{0j}$ can be found in [38, Section IV.C].

### Second algorithm to find multipole moments and the Kerr–Newman solution

In this part, we want to discuss the Bäckdahl-Herberthson algorithm for electrovacuum. The derivation has been done by Fodor, Costa Filho and Hartmann [38], in 2021, in a different way, but there is not really anything new. We can still use the recurrence relation for $y_n$ except that $y_0$ is different now. Theorem 4.17 does not even use that $\big(\widetilde{S}, \widetilde{h}\big)$ comes from a vacuum spacetime. Therefore, we can still apply this theorem and we want to apply it to both $\widetilde{\xi}$ and $\widetilde{q}$ from (7.6).

The Kerr–Newman in Boyer–Lindquist coordinates is given by

$$\begin{aligned} g = &-\left(1 - \frac{2mr - Q^2}{\rho^2}\right) dt^2 - \frac{2a(2mr - Q^2)\sin^2\theta}{\rho^2} dt d\varphi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \\ &+ \frac{\left(r^2 + a^2\right)^2 - a^2 \Delta \sin^2\theta}{\rho^2} \sin^2\theta d\varphi^2, \end{aligned} \tag{7.9}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2mr + a^2 + Q^2$. The appearing constant are a mass $m > 0$, a charge $Q \in \mathbb{R}$ and the scaled angular momentum $a \in \mathbb{R}$. For the electromagnetic field, we have $F = dA$ with

$$A = -\frac{Qr}{r^2 + a^2\cos^2\theta} dt + \frac{aQr\sin^2\theta}{\left(r^2 + a^2\cos^2\theta\right)^2} d\varphi. \tag{7.10}$$

From this potential, we can immediately read off the electric potential

$$\varphi_E = -A_t = \frac{Qr}{r^2 + a^2\cos^2\theta}. \tag{7.11}$$

A tedious calculation, again using Mathematica, shows that

$$d\varphi_B = i_\xi * F = -\frac{2aQr\cos\theta}{\left(r^2+a^2\cos^2\theta\right)^2}dr - \frac{aQ(r^2-a^2\cos^2\theta)\sin\theta}{\left(r^2+a^2\cos^2\theta\right)^2}d\theta,$$

which is integrated by

$$\varphi_B = \frac{aQ\cos\theta}{r^2+a^2\cos^2\theta}. \tag{7.12}$$

Using (2.1) we have

$$\lambda = 1 - \frac{2mr-Q^2}{r^2+a^2\cos^2\theta}.$$

For the twist one-form (2.5), a more tedious calculation using Mathematica shows that

$$\omega = \frac{2a(2mr-Q^2)\cos\theta}{\left(r^2+a^2\cos^2\theta\right)^2}dr + \frac{2a\left(mr^2-Q^2r-ma^2\cos^2\theta\right)\sin\theta}{\left(r^2+a^2\cos^2\theta\right)^2}d\theta.$$

Using $\varphi_E$ and $\varphi_B$, we find (7.5)

$$\omega^I = \frac{2aQ^2\cos\theta}{\left(r^2+a^2\cos^2\theta\right)^2}dr + \frac{2aQ^2r\sin\theta}{\left(r^2+a^2\cos^2\theta\right)^2}d\theta.$$

Therefore,

$$\omega+\omega^I = \frac{4mar\cos\theta}{\left(r^2+a^2\cos^2\theta\right)^2}dr + \frac{2ma\left(r^2-a^2\cos^2\theta\right)\sin\theta}{\left(r^2+a^2\cos^2\theta\right)^2}d\theta,$$

which is precisely the twist one-form (4.27) for the Kerr metric and it is integrated by

$$f = -\frac{2ma\cos\theta}{r^2+a^2\cos^2\theta}.$$

Using Mathematica once again, we calculate the potentials (7.6) and we find

$$\xi = \frac{m(r-m+ia\cos\theta)}{(r-m)^2+a^2\cos^2\theta} = \frac{m}{r-m-ia\cos\theta}, \tag{7.13}$$

and

$$q = \frac{Q(r-m+ia\cos\theta)}{(r-m)^2+a^2\cos^2\theta} = \frac{Q}{r-m-ia\cos\theta}. \tag{7.14}$$

The metric $h$ on the observer space, which is given by (2.2), becomes

$$\begin{aligned} h = &\frac{r^2-2mr+Q^2+a^2\cos^2\theta}{r^2-2mr+Q^2+a^2}dr^2 + \left(r^2-2mr+Q^2+a^2\cos^2\theta\right)d\theta^2 \\ &+ \left(r^2-2mr+Q^2+a^2\right)\sin^2\theta d\varphi^2. \end{aligned}$$

It is tedious to show that the Kerr–Newman spacetime is asymptotically flat, but we want to do something similar as for the Kerr spacetime in Section 4.3. Introduce a radial coordinate

$$\overline{R} = \frac{2\left(r-m-\sqrt{r^2-2mr+a^2+Q^2}\right)}{m-a^2-Q^2},$$

which is inverted by

$$r = \overline{R}^{-1}\left(1 + m\overline{R} + \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right).$$

Performing the coordinate transformation to $h$ yields

$$\begin{aligned} h = &\frac{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right)^2 - a^2\overline{R}^2\sin^2\theta}{\overline{R}^4} d\overline{R}^2 \\ &+ \frac{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right)^2 - a^2\overline{R}^2\sin^2\theta}{\overline{R}^2} d\theta^2 \\ &+ \frac{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right)^2}{\overline{R}^2} \sin^2\theta d\varphi^2. \end{aligned}$$

Like for the Kerr spacetime, we can also take the correspond conformal factor

$$\Omega(\overline{R}, \theta, \varphi) = \frac{\overline{R}^2}{\sqrt{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right)^2 - a^2\overline{R}^2\sin^2\theta}}.$$

In that case, we have

$$\widetilde{h} = \Omega^2 h = d\overline{R}^2 + \overline{R}^2 d\theta^2 + \frac{1}{1 - \frac{a^2\overline{R}^2\sin^2\theta}{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)\overline{R}^2\right)^2}} \overline{R}^2 \sin^2\theta d\varphi^2.$$

To see that all conditions Definition 3.2 hold, it is best to switch to Cartesian coordinates. However, we do not do so here. From the expression, we already recognise the flat Euclidean metric and some higher order (in $\overline{R}$) corrections in the $g_{\varphi\varphi}$-term. Moreover, we easily see that $\Omega$ and $d\Omega$ vanish at $\overline{R} = 0$ but for the second derivatives the terms were we only take derivatives of the numerator survive at $\overline{R} = 0$ and give the Euclidean metric. This should provide enough confidence that it works and if wanted, one can check it in Cartesian coordinates.

Let $\widetilde{z} = \overline{R}\cos\theta$ and $\widetilde{\rho} = \overline{R}\sin\theta$, then the metric is of the form (4.50) with

$$\gamma(\widetilde{z}, \widetilde{\rho}) = \frac{1}{2}\log\left(1 - \frac{a^2\widetilde{\rho}^2}{\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)(\widetilde{\rho}^2 + \widetilde{z}^2)\right)^2}\right).$$

This is of the wanted form and we can work through the Bäckdahl–Herberthson algorithm, even though we did not derive it from the Kerr–Newman spacetime in Weyl-Lewis-Papapetrou coordinates. The field potentials $\widetilde{\xi}$ and $\widetilde{q}$ become, using (7.13) and (7.14),

$$\widetilde{\xi}(\widetilde{z}, \widetilde{\rho}) = \frac{m\left(\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)(\widetilde{\rho}^2 + \widetilde{z}^2)\right)^2 - a^2\widetilde{\rho}^2\right)^{\frac{1}{4}}}{1 + \frac{1}{4}(m^2 - a^2 - Q^2)^2(\widetilde{\rho}^2 + \widetilde{z}^2) - ia\widetilde{z}},$$

and

$$\widetilde{q}(\widetilde{z}, \widetilde{\rho}) = \frac{Q\left(\left(1 - \frac{1}{4}(m^2 - a^2 - Q^2)(\widetilde{\rho}^2 + \widetilde{z}^2)\right)^2 - a^2\widetilde{\rho}^2\right)^{\frac{1}{4}}}{1 + \frac{1}{4}(m^2 - a^2 - Q^2)^2(\widetilde{\rho}^2 + \widetilde{z}^2) - ia\widetilde{z}}.$$

The leading order part of $\gamma$ is

$$\gamma_L(x) = \frac{1}{2}\log\bigl(1 + a^2x^2\bigr).$$

We want to change the conformal factor with a suitable $\kappa$ as in (4.58). Taking $C = 0$, (4.58) yields

$$\kappa_L(x) = \frac{1}{2}\log\bigl(1 + a^2x^2\bigr) - \log\bigl(1 - a^2x^2\bigr) = -\frac{1}{2}\log\left(\frac{\bigl(1 - a^2x^2\bigr)^2}{1 + a^2x^2}\right).$$

We have to change the conformal factor $\Omega$ correspondingly. The new conformal factor becomes $\widetilde{\Omega} = e^{\kappa}\Omega$. For the leading order parts of $\widetilde{\xi}$ and $\widetilde{q}$ with the old conformal factor, one easily calculates

$$\widetilde{\xi}_L(x) = \frac{m\bigl(1 + a^2x^2\bigr)^{\frac{1}{4}}}{1 - iax}, \qquad \widetilde{q}_L(x) = \frac{Q\bigl(1 + a^2x^2\bigr)^{\frac{1}{4}}}{1 - iax}$$

If we change the conformal factor, we get

$$e^{-\kappa_L/2}\widetilde{\xi}_L(x) = \frac{m\sqrt{1 - a^2x^2}}{1 - iax}, \qquad e^{-\kappa_L/2}\widetilde{q}_L(x) = \frac{Q\sqrt{1 - a^2x^2}}{1 - iax}.$$

For the coordinate $u$, we have

$$u = xe^{\kappa_L(x) - \gamma_L(x)} = \frac{x}{1 - a^2x^2}.$$

It is easy to verify that

$$e^{-\kappa_L/2}\widetilde{\xi}_L(x) = \frac{m}{\sqrt{1 - 2iau}}, \qquad e^{-\kappa_L/2}\widetilde{q}_L(x) = \frac{Q}{\sqrt{1 - 2iau}},$$

under this coordinate transformation. It is possible to expand this in a power expansion [7]

$$y(u) = m\sum_{k=0}^{\infty}\frac{(2k-1)!!}{k!}(iau)^k,$$

for $e^{-\kappa_L/2}\widetilde{\xi}$, and the multipole moments are found to be

$$c_k = m(ia)^k.$$

Similarly, $e^{-\kappa_L/2}\widetilde{q}$ gives

$$c_k = Q(ia)^k.$$

Therefore, the nonvanishing mass, angular momentum, electric and magnetic multipole moments are

$$m_{2k} = (-1)^k ma^{2k}, \quad j_{2k+1} = (-1)^k ma^{2k+1}, \quad e_{2k} = (-1)^k Qa^{2k}, \quad q_{2k+1} = (-1)^k Qa^{2k+1}, \tag{7.15}$$

respectively, in terms of scalars. They can be expressed in terms of tensors using (4.37).

**Theorem 7.9.** *The nonvanishing mass, angular momentum, electric and magnetic multipole moments of the Kerr–Newman solution are given by* (7.15)*, respectively, in terms of scalars. The corresponding tensors are found by* (4.37)*.*

*Proof.* The proof is given above. The multipole moments for the Kerr–Newman spacetime are also calculated in another way by Sotiriou and Apostolatos [105] in 2004. □

# Chapter 8

# Coordinate approach to multipole moments in electrovacuum

In vacuum, we defined multipole moments both in a geometric way using the Geroch–Hansen formalism and in a coordinate-dependent way using the Thorne formalism. In the previous chapter, we defined geometric multipole moments in electrovacuum, so we also want to know whether they can be read off using some suitable coordinates. Before we will do a multipole expansion for the gravitational and the electromagnetic field at the same time, we will first review multipole moments in electrostatiscs and magnetostatics in Section 8.1. Consequently, we look at multipole moments in linearised gravity in Section 8.2 and in the full nonlinear theory in Section 8.3. In this chapter, we use the notation as introduced in the beginning of Chapter 5. The discussion in this chapter is not supposed to be rigorous mathematics, but rather an argument using physical intuition.

## 8.1 Multipole moments in electrostatics and magnetostatics

In this section, we work in flat space on which we have an electric and a magnetic field, which are independent of time and there are no external sources. Then, Maxwell's equations without sources take the form

$$\nabla \times E = 0, \qquad \nabla \times B = 0, \qquad \nabla \cdot E = 0, \qquad \nabla \cdot B = 0.$$

Since the electric field is curl-free, there exists scalar potential $\varphi$ such that $E = -\nabla\varphi$. Then the fact that $E$ is also divergence-free implies that $\varphi$ is a solution of the Laplace equation. Combined with the fact that we want the electric field to decay to zero at infinity, this gives that $\varphi$ can be decomposed into spherical harmonics as

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} q^{lm} Y^{lm}(\theta, \varphi), \tag{8.1}$$

for some constants $q^{lm}$ [56]. We call the coefficients $q^{lm}$ the electric multipole moments. Then the electric field is of the form

$$E = -\nabla\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi q^{lm}}{2l+1} \frac{1}{r^{l+2}} \Big( (l+1) \mathrm{Y}^{R,lm} - \sqrt{l(l+1)}\, \mathrm{Y}^{E,lm} \Big), \tag{8.2}$$

where we recall that $\mathrm{Y}^{R,lm}$ and $\mathrm{Y}^{E,lm}$ are the pure-spin vector spherical harmonics of Definition 5.6.

We can also take another starting point. Since the electric field is divergence-free, we have $E = \nabla \times A$ for an electric vector potential $A$. We can decompose $A$ into vector spherical harmonics as

$$A = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big( A^{R,lm}(r)\mathrm{Y}^{R,lm} + A^{E,lm}(r)\mathrm{Y}^{E,lm} + A^{B,lm}(r)\mathrm{Y}^{B,lm} \Big),$$

for some functions $A^{R,lm}, A^{E,lm}, A^{B,lm}$. Using the following identities,

$$\begin{aligned}
\nabla \times \Big( f(r)\mathrm{Y}^{R,lm} \Big) &= f(r)\nabla Y^{lm} \times \mathrm{n} + Y^{lm}\nabla f \times \mathrm{n} = -\sqrt{l(l+1)}\frac{f(r)}{r}\mathrm{Y}^{B,lm}, \\
\nabla \times \Big( f(r)\mathrm{Y}^{E,lm} \Big) &= \frac{1}{\sqrt{l(l+1)}}\frac{d(rf(r))}{dr}\mathrm{n} \times \nabla Y^{lm} = \frac{1}{r}\frac{d(rf(r))}{dr}\mathrm{Y}^{B,lm}, \\
\nabla \times \Big( f(r)\mathrm{Y}^{B,lm} \Big) &= \frac{1}{\sqrt{l(l+1)}}\nabla \times \Big( f(r)\frac{\partial Y^{lm}}{\partial \theta}\widehat{\varphi} - f(r)\frac{1}{\sin\theta}\frac{\partial Y^{lm}}{\partial \varphi}\widehat{\theta} \Big) \\
&= \frac{1}{\sqrt{l(l+1)}}\Big( \frac{-l(l+1)f(r)}{r}Y^{lm}\mathrm{n} - \frac{1}{r}\frac{d(rf(r))}{dr}\frac{\partial Y^{lm}}{\partial \theta}\widehat{\theta} \\
&\qquad - \frac{1}{r\sin\theta}\frac{d(rf(r))}{dr}\frac{\partial Y^{lm}}{\partial \varphi}\widehat{\varphi} \Big) \\
&= -\sqrt{l(l+1)}\frac{f(r)}{r}\mathrm{Y}^{R,lm} - \frac{1}{r}\frac{d(rf(r))}{dr}\mathrm{Y}^{E,lm},
\end{aligned}$$

we see that the electric field is given by

$$\begin{aligned}
E &= \nabla \times A \\
&= \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big( -\sqrt{l(l+1)}\frac{A^{R,lm}(r)}{r}\mathrm{Y}^{B,lm} + \frac{1}{r}\frac{d(rA^{E,lm}(r))}{dr}\mathrm{Y}^{B,lm} \\
&\qquad -\sqrt{l(l+1)}\frac{A^{B,lm}(r)}{r}\mathrm{Y}^{R,lm} - \frac{1}{r}\frac{d(rA^{B,lm}(r))}{dr}\mathrm{Y}^{E,lm} \Big).
\end{aligned} \tag{8.3}$$

Applying the curl once more gives

$$\begin{aligned}
\nabla \times E = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \Big( & l(l+1)\frac{A^{B,lm}(r)}{r^2}\mathrm{Y}^{B,lm} - \frac{1}{r}\frac{d^2(rA^{B,lm}(r))}{dr^2}\mathrm{Y}^{B,lm} \\
& +l(l+1)\frac{A^{R,lm}(r)}{r^2}\mathrm{Y}^{R,lm} + \sqrt{l(l+1)}\frac{1}{r}\frac{dA^{R,lm}(r)}{dr}\mathrm{Y}^{E,lm} \\
& -\sqrt{l(l+1)}\frac{1}{r^2}\frac{d(rA^{E,lm}(r))}{dr}\mathrm{Y}^{R,lm} - \frac{1}{r}\frac{d^2(rA^{E,lm}(r))}{dr^2}\mathrm{Y}^{E,lm} \Big).
\end{aligned}$$

But the electric field is curl-free, so we must have $A^{B,lm}(r) \sim \frac{1}{r^{l+1}}$ and $\sqrt{l(l+1)}A^{R,lm}(r) = \frac{d(rA^{E,lm})}{dr}$ [2]. Introduce constants $Q^{lm}$ such that $A^{B,lm}(r) = \frac{Q^{lm}}{r^{l+1}}$, then we see that the vector potential is

$$A = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \left( \frac{1}{\sqrt{l(l+1)}}\frac{d\big(rA^{E,lm}(r)\big)}{dr}\mathrm{Y}^{R,lm} + A^{E,lm}(r)\mathrm{Y}^{E,lm} + \frac{Q^{lm}}{r^{l+1}}\mathrm{Y}^{B,lm} \right), \tag{8.4}$$

and we interpret the constants $Q^{lm}$ as multipole moments. Substituting (8.4) into (8.3), we see that the electric field is

$$E = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{Q^{lm}}{r^{l+2}}\Big(-\sqrt{l(l+1)}\mathrm{Y}^{R,lm} + l\mathrm{Y}^{E,lm}\Big), \tag{8.5}$$

So, the electric field does not feel $A^{R,lm}(r)$ and $A^{E,lm}(r)$ but it only feels the multipole moments $Q^{lm}$ via $A^{B,lm}(r)$. Clearly, we want the two expressions (8.2) and (8.5) to coincide, so

$$Q^{lm} = -\frac{4\pi q^{lm}}{2l+1}\sqrt{\frac{l+1}{l}}.$$

From this relation it is clear that both scalar and vector potentials can be used to define multipole moments and the resulting sets of multipole moments are equivalent. Note that there is freedom left in the vector potential (8.4): the function $A^{E,lm}(r)$. This is as expected because these terms can be written as a total gradient and therefore represent the gauge freedom:

$$\frac{1}{\sqrt{l(l+1)}}\frac{d\big(rA^{E,lm}(r)\big)}{dr}\mathrm{Y}^{R,lm} + A^{E,lm}(r)\mathrm{Y}^{E,lm} = \nabla\Big(rA^{E,lm}(r)Y^{lm}\Big).$$

Without sources, Maxwell's equations are symmetric in $E$ and $B$. So, for the magnetic field we have exactly the same decompositions. In particular, one may want to decompose the four-potential $A = -\varphi dt + A_i dx^i$, where $\varphi$ is decomposed using (8.1) and $(A_1, A_2, A_3)$ is decomposed using (8.4). Then the electromagnetic field tensor $F$ is decomposed by decomposing the electric field using the scalar potential and the magnetic field using the vector potential. Equivalently, we can also decompose the dual four-potential $\widetilde{A}$ with the magnetic scalar potential and the electric vector potential. We summarise this discussion with the following definition:

**Definition 8.1.** Let $\varphi_E$ be the electric scalar potential and let $\varphi_B$ be the magnetic scalar potential without sources such that they decay to zero at infinity, then the *electric multipole moments* $q^{lm}$ and the *magnetic multipole moments* $b^{lm}$ are of the form

$$\varphi_E = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{1}{r^{l+1}} q^{lm} Y^{lm}(\theta,\varphi),$$

and

$$\varphi_B = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{4\pi}{2l+1}\frac{1}{r^{l+1}} b^{lm} Y^{lm}(\theta,\varphi).$$

Equivalently, they can also be read off as the parity $\pi = (-1)^{l+1}$ part of the electric vector potential $A_E$ and the magnetic vector potential $A_B$ which are given by

$$A_E = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_E^{R,lm}(r)\mathrm{Y}^{R,lm} + A_E^{E,lm}(r)\mathrm{Y}^{E,lm} - \frac{4\pi}{2l+1}\sqrt{\frac{l+1}{l}}\frac{q^{lm}}{r^{l+1}}\mathrm{Y}^{B,lm}\right),$$

and

$$A_B = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\left(A_B^{R,lm}(r)\mathrm{Y}^{R,lm} + A_B^{E,lm}(r)\mathrm{Y}^{E,lm} - \frac{4\pi}{2l+1}\sqrt{\frac{l+1}{l}}\frac{b^{lm}}{r^{l+1}}\mathrm{Y}^{B,lm}\right),$$

where $\sqrt{l(l+1)}A_E^{R,lm}(r) = \frac{d(rA_E^{E,lm})}{dr}$ and $\sqrt{l(l+1)}A_B^{R,lm}(r) = \frac{d(rA_B^{E,lm})}{dr}$ for some functions $A_E^{E,lm}$ and $A_B^{E,lm}$.

Like we discussed in Section 5.1, there is a one-to-one correspondence between symmetric trace-free tensors and spherical harmonics. We can express the multipole moments in terms of symmetric trace-free tensors using (5.1) and (5.6). For the scalar potential, this gives

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \frac{1}{r^{l+1}} \mathcal{Q}_{A_l} N_{A_l}, \tag{8.6}$$

where $\mathcal{Q}$ is a symmetric trace-free tensor such that $\mathcal{Q}_{A_l} = \sum_{m=-l}^{l} q^{lm} \mathcal{Y}_{A_l}^{lm}$. Here, $\mathcal{Y}_{A_l}^{lm}$ is given by (5.2). For the vector potential, the expression is a bit ugly as there is a lot of freedom in $A^{E,lm}(r)$. Nevertheless, it must be of the form

$$A = \sum_{l=0}^{\infty} \left( \sum_{m=-l}^{l} \frac{1}{r^{l+1}} \frac{4\pi}{2l+1} \mathcal{Q}_{A_l} N_{A_l} + \Big( l \text{ pole with parity } \pi = (-1)^l \Big) \right), \tag{8.7}$$

## 8.2 Linearised Einstein–Maxwell solutions

In Section 5.2, we decomposed the metric tensor in spherical harmonics for vacuum solutions of the Einstein equations. In the previous section, we decomposed the electric and magnetic field in spherical harmonics in flat space. Now, we want to combine the two approaches and decompose the gravitational and electromagnetic fields simultaneously when they are coupled via the Einstein–Maxwell equations. To do so, we start with linearised gravity. Like in the Thorne formalism, all indices are raised and lowered using the Minkowski metric rather than the full metric tensor.

Suppose $M$ is diffeomorphic to $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \overline{\mathbb{B}^3}\right)$ and write

$$g_{\mu\nu} = \eta_{\mu\nu} + g^1_{\mu\nu}.$$

Let $\gamma^1_{\mu\nu}$ be the trace-reverse of $g^1_{\mu\nu}$, i.e.,

$$\gamma^1_{\mu\nu} = g^1_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma} g^1_{\rho\sigma}.$$

In the Lorenz gauge, which is especially useful for the linearised gravity [25, 53], we have $\partial_\nu \gamma^{1\mu\nu} = 0$. If the metric components are independent of the time-coordinate, this gives $\partial_j \gamma^{1\alpha j} = 0$. The linearised Einstein equations in this gauge reduce to

$$\Box \gamma^{1\mu\nu} = -16\pi T^{\mu\nu}.$$

Again, since there is no time-dependence, we can replace the flat wave operator $\Box$ by the flat Laplacian $\Delta$.

In Minkowski spacetime, we have $T_{00} = |E|^2 + |B|^2$, where $E$ is the electric field and $B$ is the magnetic field determined from $F$ through

$$E = -i_\xi F, \qquad B = -i_\xi * F,$$

where $\xi = \frac{\partial}{\partial x^0}$. The Einstein–Maxwell equations imply that the electric and magnetic fields must vanish at leading order, so $F$ also vanishes at leading order. Therefore, we want to expand the electromagnetic field around zero, just as we want to expand the metric around the Minkowski metric. Since the stress-energy tensor is quadratic in $F$, this implies that it vanishes when only considering the first-order term [112, 117].

Hence, up to first order, $\gamma^{1\mu\nu}$ must solve the Laplace equations. Therefore, we can express the components in terms of spherical harmonics as in Section 5.1. We want to have the same factors as in vacuum, in which case we get [107, Equation (8.12)]

$$\gamma^1{}_{00} = \frac{4\mathcal{M}}{r} + \sum_{l=2}^{\infty}(-1)^l \frac{4}{l!}\mathcal{M}_{A_l}\left(r^{-1}\right)_{,A_l}, \tag{8.8a}$$

$$\gamma^1{}_{0j} = \frac{-2\epsilon_{jpq}\mathcal{J}_p n_q}{r^2} - \sum_{l=2}^{\infty}(-1)^l \frac{4l}{(l+1)!}\epsilon_{jpq}\mathcal{J}_{pA_{l-1}}\left(r^{-1}\right)_{,qA_{l-1}}, \tag{8.8b}$$

$$\gamma^1{}_{ij} = 0. \tag{8.8c}$$

Here, $\mathcal{M}_{A_l}$ and $\mathcal{J}_{A_l}$ are constants representing the multipole moments like in vacuum. Reversing the trace again and adding the Minkowski metric gives

$$\begin{aligned}
g_{00} &= -1 + \frac{2\mathcal{M}}{r} + \sum_{l=2}^{\infty}(-1)^l \frac{2}{l!}\mathcal{M}_{A_l}\left(r^{-1}\right)_{,A_l}, \\
g_{0j} &= \frac{-2\epsilon_{jpq}\mathcal{J}_p n_q}{r^2} - \sum_{l=2}^{\infty}(-1)^l \frac{4l}{(l+1)!}\epsilon_{jpq}\mathcal{J}_{pA_{l-1}}\left(r^{-1}\right)_{,qA_{l-1}}, \\
g_{ij} &= \delta_{ij}\left(1 + \frac{2\mathcal{M}}{r} + \sum_{l=2}^{\infty}(-1)^l \frac{2}{l!}\mathcal{M}_{A_l}\left(r^{-1}\right)_{,A_l}\right).
\end{aligned}$$

There is no mass dipole moment because we take coordinates that are mass-centered.

We also want to linearise the electromagnetic field. Then, $F$ is determined by $dF = 0$ and $d * F = 0$. Suppose now that $F$ is exact with an electromagnetic potential $A$ that is also stationary. Then we must have $d * dA = 0$. In the Lorenz gauge, we have $d * A = 0$, so we see that $\Box_g^H A = 0$, where $\Box_g^H$ denotes the Hodge Laplacian. Linearising this equation gives $\Box A_\mu = 0$ with respect to the flat wave operator. Since the components of the electromagnetic potential do not depend on $t$, they satisfy the Laplace equation. Hence, we can decompose the vector potential as in Section 8.1, defining electromagnetic multipole moments.

Alternatively, we saw in Proposition 7.3 that there exist electric and magnetic scalar potentials $\varphi_E$ and $\varphi_B$, respectively, under reasonable assumptions. They are defined by $i_\xi F = d\varphi$ and $i_\xi * F = d\psi$, where $\xi = \frac{\partial}{\partial x^0}$. In our coordinates, we have $\xi_\mu = g_{0\mu}$, so at zeroth order we have $\xi^\flat = -dt$. Since we are only interesting in the first order corrections, we can observe that $dF = 0$ implies that $d * i_\xi * F = 0$ and $d * F = 0$ implies that $d * i_\xi F = 0$. Therefore, $\varphi_E$ and $\varphi_B$ satisfy $\Box_g^H \varphi = 0$ to first order, showing that $\varphi_E$ and $\varphi_B$ are solutions of the Laplace equation to first order as they are independent of time. Hence, we can also apply to method of Section 8.1 to determine equivalent electromagnetic multipole moments based on the electric and magnetic scalar potentials.

The easiest (and trivial) example in which we can apply this construction is the Reissner–Nordström spacetime. The metric is given by

$$g = -\left(1 - \frac{2mr - Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2mr - Q^2}{r^2}\right)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta d\varphi^2\big),$$

and the electromagnetic potential is

$$A = -\frac{Q}{r}dt.$$

We see that both $g$ and $A$ are spherically symmetric. From $A$, we get the electromagnetic scalar potentials $\varphi_E = \frac{Q}{r}$ and $\varphi_B = 0$. Hence, we can immediately observe that the charge of the system is $Q$ and all other multipole moments vanish. Since the metric is static, i.e., $g_{0j} = 0$, we can also read off immediately that the angular momentum multipole moments vanish. For the mass multipole moments, observe that $\frac{Q^2}{r^2}$ is already a second order term because we also only consider linear perturbations of the electromagnetic field. Therefore,

$$g_{00} = -1 + \frac{2m}{r},$$

to first order. Therefore, we find the mass monopole moment to be $m$. Note that we assigned the same weight to $m$ and $q$. They also have the same geometrised units so that is quite reasonable. It would be way more interesting to consider the Kerr–Newman spacetime, but that is better done in the next section.

## 8.3 Multipole moments in curved electrovacuum

In the previous section, we discussed linear perturbations of the gravitational and electromagnetic fields. However, we are also interested in nonlinear perturbations. The goal of this section is to discuss them.

Recall the metric density $\mathfrak{g}^{\mu\nu} = \sqrt{-\det g}g^{\mu\nu}$ from Section 5.2 and that the harmonic gauge condition reads $\partial_\beta \mathfrak{g}^{\alpha\beta} = 0$. Define

$$\overline{h}^{\mu\nu} = \eta^{\mu\nu} - \mathfrak{g}^{\mu\nu}.$$

Then we can also write the harmonic gauge condition as $\partial_\beta \overline{h}^{\alpha\beta} = 0$. In harmonic coordinates, the Einstein equations read

$$\Box \overline{h}^{\alpha\beta} = W^{\alpha\beta} = -16\pi(-\det g)\Big(T^{\alpha\beta} + t^{\alpha\beta}_{LL} + t^{\alpha\beta}_{H}\Big).$$

Here, $t^{\alpha\beta}_{LL}$ is the (harmonic) Landau–Lifshitz pseudotensor and it is given by

$$\begin{aligned} 16\pi(-\det g)t^{\alpha\beta}_{LL} = {} & \frac{1}{2}g^{\alpha\beta}g_{\rho\sigma}\partial_\mu \mathfrak{g}^{\rho\nu}\partial_\nu \mathfrak{g}^{\mu\sigma} - g^{\alpha\nu}g_{\rho\sigma}\partial_\mu \mathfrak{g}^{\beta\rho}\partial_\nu \mathfrak{g}^{\mu\sigma} \\ & - g^{\beta\mu}g_{\rho\sigma}\partial_\mu \mathfrak{g}^{\nu\rho}\partial_\nu \mathfrak{g}^{\alpha\sigma} + g_{\rho\sigma}g^{\mu\nu}\partial_\mu \mathfrak{g}^{\beta\rho}\partial_\nu \mathfrak{g}^{\alpha\sigma} \\ & + \frac{1}{8}\Big(2g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu}\Big)(2g_{\rho\lambda}g_{\sigma\tau} - g_{\rho\sigma}g_{\tau\lambda})\partial_\mu \mathfrak{g}^{\rho\sigma}\partial_\nu \mathfrak{g}^{\tau\lambda}. \end{aligned}$$

Thorne and Kovács provide a way to express the Landau–Lifshitz pseudotensor in terms of a power expansion in $\overline{h}_{\alpha\beta}$ [108]. The other term is typical for the harmonic gauge and is given by

$$16\pi(-\det g)t_H^{\alpha\beta} = -\overline{h}^{\mu\nu}\partial_{\mu\nu}\overline{h}^{\alpha\beta} + \partial_\mu\overline{h}^{\alpha\nu}\partial_\nu\overline{h}^{\beta\mu}.$$

We expand the gravitational field $\overline{h}_{\alpha\beta}$ as

$$\overline{h}_{\alpha\beta} = \sum_{p=1}^{\infty} G^p \gamma^p_{\alpha\beta},$$

where $G$ serves as a bookkeeping device in the expansion as we set Newton's constant of gravity to one. When gravity is turned off, that is when $g_{\alpha\beta} = \eta_{\alpha\beta}$, also electromagnetism is turned off. Therefore, we also write

$$F_{\alpha\beta} = \sum_{p=1}^{\infty} G^p f^p_{\alpha\beta}.$$

Finally, we expand $W_{\alpha\beta}$ as

$$W_{\alpha\beta} = \sum_{p=2}^{\infty} G^p w^p_{\alpha\beta},$$

which only starts at $p = 2$ because the stress-energy tensor is second order in $F_{\alpha\beta}$ and $t_{LL}^{\alpha\beta}$ and $t_H^{\alpha\beta}$ are second order in $\overline{h}_{\alpha\beta}$. Still, we assume all these functions are independent of $t$. Then the harmonic gauge condition becomes

$$\gamma^p_{\alpha j,j} = 0 \tag{8.9}$$

and the Einstein equations becomes

$$\Box\gamma^p_{\alpha\beta} = w^p_{\alpha\beta}. \tag{8.10}$$

For the electromagnetic field, we also want $dF = 0$ and $*d*F = 0$. The first equation easily translates to

$$f^p_{\alpha\beta,\gamma} + f^p_{\beta\gamma,\alpha} + f^p_{\gamma\alpha,\beta} = 0. \tag{8.11}$$

The second equation is more subtle. It can equivalently be written as

$$g^{\beta\gamma}\nabla_\beta F_{\gamma\alpha} = 0, \tag{8.12}$$

but not only powers of $G$ are contained in $F_{\alpha\beta}$ but also in $g^{\alpha\beta}$ and the Christoffel symbols. Using the relation between the metric tensor and $\overline{h}$, it is possible to express the metric tensor as a power series in terms of $\overline{h}$ [108]. If we expand the left-hand side of (8.12) in $G$, we see the $G^p$-term only contains $\gamma^q_{\alpha\beta}$ and $f^q_{\alpha\beta}$ for $q \le p$. For $p = 1$, we do not need to take the correction terms with $\gamma^q_{\alpha\beta}$ into account and we simply have

$$\eta^{\beta\gamma}\partial_\beta f^1_{\gamma\alpha} = 0. \tag{8.13}$$

So, we want to find $\gamma^1_{\alpha\beta}$ and $f^1_{\alpha\beta}$ solving (8.9), (8.10), (8.11) and (8.13) for $p = 1$ with $w^1_{\alpha\beta} = 0$. This is exactly the same as in the linearised case. Equation (8.9) corresponds to the Lorenz gauge, equation (8.10) for $p = 1$ corresponds to the linearised Einstein equation and equations (8.11) and (8.13) for $p = 1$ reduce just Maxwell's equations.

For $p > 1$, we want to analyze $w^p_{\alpha\beta}$. We can express the metric tensor, its determinant, the Landau-Lifshitz pseudotensor and $t^{\alpha\beta}_H$ in terms of a power expansion in $\overline{h}_{\alpha\beta}$. Hence, $w^p_{\alpha\beta}$ is a sum of terms that consist of products of (derivatives of) $\gamma^q_{\mu\nu}$ and $f^q_{\mu\nu}$ with $q \leq p - 1$ because we have the product of at least two functions. So if we know $\gamma^q_{\alpha\beta}$ and $f^q_{\alpha\beta}$ for $q \leq p - 1$, we know the right-hand side of equation (8.10). The $G^p$-term in equation (8.12) consists of a term of the form $\eta^{\beta\gamma}\partial_\beta f^p_{\gamma\alpha}$ like for $p = 1$ and all the other terms only contain $\gamma^q_{\alpha\beta}$ and $f^q_{\alpha\beta}$ for $q \leq p - 1$. These lower order terms in a sense act like a source term in Maxwell's equations. In any case, we can also see this as an equation for $f^p_{\alpha\beta}$ where all other terms are given. So, the field equations of order $p$ are generated by $\gamma^q_{\alpha\beta}$ and $f^q_{\alpha\beta}$ for $q \leq p - 1$ and determine $\gamma^p_{\alpha\beta}$ and $f^p_{\alpha\beta}$. These functions can thus be determined by solving the field equations recursively.

Up to now, everything works as long as we assume everything converges properly. The next step is to introduce some characteristic length scales. In Section 8.2, we introduced the mass $\mathcal{M}$ up to linear order. Similarly, we also have the charge $\mathcal{Q}$ up to linear order. Let $M = \max\{\mathcal{M}, \mathcal{Q}\}$ be the mass scale. Note that it means that we assign the same weight to mass and charge. The length scale $R$ measures the nonspherical deformations of the source's gravitational and electromagnetic fields and is defined as

$$R = \max_{l\geq 1}\Big\{|\mathcal{M}_{A_l}/M|^{1/l}, |\mathcal{J}_{A_l}/M|^{1/l}, |\mathcal{Q}_{A_l}/M|^{1/l}, |\mathcal{B}_{A_l}/M|^{1/l}\Big\}.$$

Hence, the gravitational and electromagnetic $2^l$-pole moments are bounded by $MR^l$. If the source is of size $L$, it is reasonable to expect that $R \leq L$ and we are interested in the region far away from the source, meaning $r \gg M, Q$ and $r > L \geq R$. We expand $\overline{h}_{\mu\nu}$, $F_{\mu\nu}$, $W_{\mu\nu}$, and $g_{\mu\nu}$ in a series where each term scales as powers of $M$ and $R$ and some order of spherical harmonics. We write

$$\overline{h}_{\mu\nu} = \sum_{p,n,l} \gamma^{pnl}_{\mu\nu}, \qquad F_{\mu\nu} = \sum_{p,n,l} f^{pnl}_{\mu\nu}, \qquad W_{\mu\nu} = \sum_{p,n,l} w^{pnl}_{\mu\nu}, \qquad g_{\mu\nu} = \sum_{p,n,l} g^{pnl}_{\mu\nu},$$

where $\gamma^{pnl}_{\mu\nu}$ scales as $M^pR^n$ and only contains spherical harmonics of order $l$, and similarly for the other decompositions. Sometimes, we want to sum over $p$, $n$ or $l$. In that case, we replace the corresponding index/indices by dot(s). For example, $\gamma^{pn\cdot}_{\mu\nu} = \sum_l \gamma^{pnl}_{\mu\nu}$. Since the monopole moments scale as $M$ and $\gamma^1_{\mu\nu}$ and $f^1_{\mu\nu}$ in the nonlinear expansion equal the linear perturbations, we see that $\gamma^1_{\mu\nu}$ and $f^1_{\mu\nu}$ scale as $M$. Inductively, we see that $\gamma^p_{\mu\nu}$ and $f^p_{\mu\nu}$ scale as $M^p$ using equations (8.10) and (8.12), respectively. Therefore,

$$\gamma^p_{\mu\nu} = \gamma^{p\cdot\cdot}_{\mu\nu}, \qquad f^p_{\mu\nu} = f^{p\cdot\cdot}_{\mu\nu}.$$

The same holds when decomposing the metric tensor and $W$ as we see by expressing them in terms of $\gamma^p_{\mu\nu}$ and $f^p_{\mu\nu}$. From (8.8) it is clear that the only nonzero parts of $\gamma^1_{\mu\nu}$ are $\gamma^{100}_{00}$, $\gamma^{1ll}_{00}$ for $l \geq 2$, and $\gamma^{1ll}_{0j}$ for $l \geq 1$.

So, we do not only want to decompose equations (8.9), (8.10), (8.11), and (8.12) in orders of $M$, but also in orders of $R$ and in the order of spherical harmonics. The harmonic gauge condition becomes simply

$$\gamma^{pnl}_{\alpha j,j} = 0.$$

For the Einstein equation, we have

$$\Delta \gamma^{pnl}_{\mu\nu} = w^{pnl}_{\mu\nu}. \tag{8.14}$$

The object $W^{\mu\nu}$ satisfies the conservation relation [91, Equation (6.54)]

$$\partial_\nu W^{\mu\nu} = 0.$$

Since the components are independent of time, this translates to the condition

$$w^{pnl}_{\alpha j,j} = 0.$$

For Maxwell's equations, we find

$$f^{pnl}_{\mu\nu,\rho} + f^{pnl}_{\nu\rho,\mu} + f^{pnl}_{\rho\mu,\nu} = 0,$$

and the other equation is more subtle, but still gives a differential equation for $f^{pnl}_{\mu\nu}$ when we know all mass orders $q \leq p-1$. Knowing the linear terms, i.e., the terms with $p = 1$, the higher order terms can still be calculated recursively. The components $\gamma_{\mu\nu}$ are determined by their Laplacian and divergence. We assume that we found a particular solution, but we may need to add a homomogeneous solution. Homogeneous solutions of the Laplace equation behave like $r^{l'}$ or $r^{-(l'+1)}$ with $l' \geq 0$. Solutions of the latter type could have been included in the linear part, but solutions of the former type may appear. Since we want the homogeneous part of $\gamma^{pnl}_{\mu\nu}$ to be dimensionless, we want $-p-n = l'$. But $-p-n < 0 \leq l'$, which is a contradiction, so we do not allow homogeneous pieces. If we would have allowed for time dependence, there can be homogeneous pieces [107]. The electromagnetic field is determined by their divergence and exterior derivative. If we assume the electromagnetic field is exact, then equation (8.11) is automatically satisfied and we can replace equation (8.12) by a Poisson equation. In that case, we can apply a similar reasoning as for $\gamma^{pnl}_{\mu\nu}$ to conclude there are no homogeneous pieces. If we do not assume exactness, it is a bit more difficult. However, like in Section 7.1, we do have scalar electric and magnetic potentials. In that case, we have multipole the multipole expansion (8.1) for both $\varphi_E$ and $\varphi_B$ and we find an expansion for the electromagnetic field tensor using (7.3).

With arbitrary time dependence, logarithmic terms can appear in the perturbations [107]. However, if the spacetime is stationary, that is not possible. Using induction, the form of $w^{pnl}_{\mu\nu}$ in terms of the other quantities, and equation (8.14), one can show that again no logarithmic terms will appear in the same way as in Thorne [107, Sections IX and X]. Moreover, there cannot by "tail terms" as discussed in Thorne [107, Appendix]. Therefore, the perturbations $g^{pnl}_{\mu\nu}$ must be sums of terms of the form

$$g^{pnl}_{\mu\nu} = \left( \prod_{j=1}^{p} \mathcal{M}^{l_j \pi_j} \right)^{l\pi} r^{-p-n}.$$

The laws of angular momentum coupling require that $\sum_{j=1}^{p} l_j = n \geq l \geq 0$, $\min \left| \sum_{j=1}^{p} \pm l_j \right| \leq l$. Moreover, $\pi = \pi_1 \cdots \pi_p$, $(\cdot)^{l\pi}$ means taking the spherical harmonic of order $l$ and parity $\pi$, and

$$\mathcal{M}^{l_j \pi_j} = \begin{cases} \mathcal{M}_{A_{l_j}}, & \pi_j = (-1)^{l_j}, \\ \epsilon_{ipq} \mathcal{J}_{p A_{l_{j-1}}} n_q, & \pi = (-1)^{l_j+1}. \end{cases}$$

In mass-centered coordinates, this gives precisely the metric as in (5.12). Therefore, we can still define the gravitational multipole moments in the same way, even with the presence of an electromagnetic field.

We also want electromagnetic multipole moments. It is again easier to work with the potentials than with the field tensor because the differential equations are easier to handle with. Write $\varphi = \sum_{p,n,l} \varphi^{pnl}$ for the potentials as above. Then Maxwell's equations imply that

$$\Delta\varphi^{pnl} = v^{pnl},$$

where $v^{pnl}$ consists of products of lower order terms. Therefore, we can solve for $\varphi^{pnl}$ when we know all orders $q \leq p-1$. Similar considerations as above show that

$$\varphi = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{r^{l+1}} \left( \frac{4\pi}{2l+1} \mathcal{Q}_{A_l} N_{A_l} + S_{l-1} \right). \tag{8.15}$$

We can still define ACMC coordinates like in Definition 5.15. It seems almost certain to me that Theorem 5.16 still holds, which would make the multipole moments well-defined. We do not delve into this here. We also expect Theorem 6.4 still holds, meaning the multipole moments introduced here are equivalent to the ones in Definition 7.6. Since the metric has the same form as in the Thorne formalism, we can almost follow the proof of Theorem 6.4 in verbatim to conclude the result must hold for the gravitational multipole moments. If we pick the conformal factor of the form

$$\Omega = \frac{1}{r} + \sum_{l=2}^{\infty} \frac{S_{l-1}}{r^{l+1}}, \tag{8.16}$$

as we did in the proof, then the highest order poles in the electromagnetic potentials also will not change when dividing by $\Omega^{\frac{1}{2}}$. Therefore, we expect Theorem 6.4 and we would have to find the proportionality constant for the electromagnetic multipole moments.

We rather end this chapter with a quick look at the Kerr–Newman metric. Recall that the metric and electromagnetic potential are given by (7.9) and (7.10). Like for the Kerr metric in Section 5.3, Boyer–Lindquist coordinates are ACMC-0 and we have, after normalising the metric,

$$\begin{aligned}
g_{tt} &= -1 + \frac{2m}{r} + \frac{Q^2}{r^2} + O(r^{-3}), \\
g_{t\varphi} &= \frac{4ma\sin\theta}{r^2} + O(r^{-3}), \\
g_{rr} &= 1 + \frac{2m}{r} + \frac{4m^2 - Q^2 - a^2\sin^2\theta}{r^2} + O(r^{-3}), \\
g_{\theta\theta} &= 1 + \frac{a^2\cos^2\theta}{r^2}, \\
g_{\varphi\varphi} &= 1 + \frac{a^2}{r^2} + O(r^{-3}).
\end{aligned}$$

Like for Kerr, we find that $\mathcal{M} = m$, $\mathcal{M}_a = 0$, $\mathcal{J} = 0$ and $\mathcal{J}_z = ma$ while $\mathcal{J}_x = \mathcal{J}_y = 0$. Moreover, the electromagnetic scalar potentials are given by (7.11) and (7.12) and we find

$$\varphi_E = \frac{Q}{r} + O(r^{-3}),$$
$$\varphi_B = \frac{aQ\cos\theta}{r^2} + O(r^{-4}).$$

Let $\mathcal{Q}_{A_l}$ denote the electric multipole moments and $\mathcal{B}_{A_l}$ the magnetic multipole moments, then these equations show that $\mathcal{Q} = \frac{Q}{4\pi}$, $\mathcal{Q}_a = 0$, $\mathcal{B} = 0$ and $\mathcal{B}_z = \frac{3aQ}{4\pi}$ while $\mathcal{B}_x = \mathcal{B}_y = 0$. This is precisely what we want when comparing to Theorem 7.9 up to proportionality. The constants are a bit awkward, but we can rescale the multipole moments if we want.

We can also use harmonic coordinates. In 2014, Lin and Jiang [75] found harmonic coordinates preserving the asymptotically flat form. Using Mathematica, I have been able to calculate the first 11 multipole moments with these coordinates and they are indeed proportional to the ones found in Theorem 7.9.

# Chapter 9

# Multipole moments for other matter fields

In this chapter, we want to generalise multipole moments to broader classes of spacetimes. That is, we want to extend them to non-electrovacuum solutions of the Einstein equations. We start with a discussion on arbitrary matter fields in Section 9.1. To illustrate another class of solutions in which we can define multipole moments, we consider scalar field solutions in Section 9.2.

## 9.1 Gravitational multipole moments with arbitrary matter fields

The approach we consider to define gravitational multipole moments in presence of arbitrary matter fields is due to Mayerson [77]. Like in Section 7.1, the main problem when carrying out the Geroch-Hansen formalism is that the twist one-form is not necessarily closed. Therefore, we want to define an improved twist one-form that is closed, and then we can carry out the Geroch-Hansen formalism again.

Again, we assume our spacetime $(M, g)$ is stationary with a stationary vector field $\xi$. Recall from Theorem 2.12 that we have a twist one-form $\omega$ whose exterior derivative is

$$d\omega = 2i_\xi * Rc(\xi, \cdot) = -2 * \Big(\xi^\flat \wedge Rc(\xi, \cdot)\Big). \tag{9.1}$$

Let $T$ be the stress-energy tensor, then the Einstein equations imply that

$$Rc - \frac{1}{2}gR = T.$$

Define a one-form $\alpha$ on $M$ as

$$\alpha = T(\xi, \cdot) + \lambda^{-1}T(\xi, \xi)\xi^\flat. \tag{9.2}$$

By the Einstein equations, we have $\mathcal{L}_\xi T = 0$ because $\xi$ is a Killing vector and we have $\alpha(\xi) = 0$. We want to calculate the divergence of $\alpha$. By the conservation law for the stress-energy tensor, we have $\nabla^\mu T_{\mu\nu} = 0$. Moreover, the total covariant derivative of $\xi^\flat$ is antisymmetric because

$\xi$ is a Killing vector, and the stress-energy tensor is symmetric. Since $\mathcal{L}_\xi T = 0$ and $\mathcal{L}_\xi \xi = 0$, we see that $\mathcal{L}_\xi T(\xi, \xi) = \nabla_\xi T(\xi, \xi) = 0$. Therefore,

$$\begin{aligned} \nabla^\mu \alpha_\mu &= (\nabla^\mu T_{\mu\nu})\xi^\nu + T_{\mu\nu}\nabla^\mu \xi^\nu + 2\lambda^{-2}\xi_\nu(\nabla^\mu \xi^\nu)T(\xi,\xi)\xi_\mu \\ &\quad + \lambda^{-1}(\nabla^\mu T(\xi,\xi))\xi_\mu + \lambda^{-1}T(\xi,\xi)\nabla^\mu \xi_\mu \\ &= 0. \end{aligned}$$

Since the divergence of $\alpha$ vanishes, we have $d * \alpha = 0$. Define the two-form $\eta$ on $M$ by

$$\eta = i_\xi * \alpha. \tag{9.3}$$

Then we have, using Cartan's magic formula and the fact that $\mathcal{L}_\xi$ commutes with $*$ because $\xi$ is a Killing vector field,

$$d\eta = di_\xi * \alpha = \mathcal{L}_\xi * \alpha - i_\xi d * \alpha = 0,$$

so $\eta$ is closed. Moreover,

$$\mathcal{L}_\xi \eta = \mathcal{L}_\xi i_\xi * \alpha = i_\xi \mathcal{L}_\xi * \alpha = i_\xi * \mathcal{L}_\xi \alpha = 0,$$

and

$$i_\xi \eta = i_\xi i_\xi * \alpha = 0.$$

Therefore, $\eta$ is a two-form that lives on $S$, and it is also closed on $S$. However, we want $\eta$ to be an exact one-form on $S$. That is a condition that is typically not true. However, it is also not very unreasonable to expect $\eta$ to be exact. As usual, we assume $S$ is diffeomorphic to $\mathbb{R}^3 \setminus \overline{\mathbb{B}}^3$, in which case the second de Rham cohomology is $\mathbb{R}$. In other words, up to scalar multiples and adding exact two-forms, there is only one closed two-form that is not exact. We use the diffeomorphism between $S$ and $\mathbb{R}^3 \setminus \overline{\mathbb{B}}^3$ as a coordinate chart for $S$, and we use spherical coordinates $(r, \theta, \varphi)$. Let $S_r$ denote the sphere in $S$ of radius $r$ for $r > 1$ using this chart. Then $\eta$ is an exact two-form if and only if its integral over one, and hence all, of the $S_r$'s vanishes. With a time coordinate along the stationary vector field, we see that $\eta$ restricted to $S_r$ must look like $\sqrt{-\det g}\Big(T_{0r} - \frac{g_{0r}}{g_{00}}T_{00}\Big)d\theta \wedge d\varphi$ because there cannot be a time-component. If the metric would have been the Minkowski metric, we have $\sqrt{-\det g} = r^2 \sin\theta$ and $g_{0r} = 0$. So, to first order, we have

$$\int_{S_r} \eta \sim \int_0^\pi d\theta \int_0^{2\pi} d\varphi T_{0r} r^2 \sin\theta. \tag{9.4}$$

If the metric is of the form as in the Thorne formalism, we easily observe using the Einstein equations that $T_{0r} = O(r^{-3})$. If we assume this as well, we see the integral in (9.4) vanishes in the limit $r \to \infty$. So, for large $r$, we expect $\int_{S_r} \eta$ to be close to zero. If it is zero for some $r$, $\eta$ is exact.

Suppose $\eta$ is an exact 2-form on $S$, then there is a one-form $\beta$ on $S$ such that $\eta = d\beta$. Let $\omega^I \in \Omega^1(S)$ be given by $\omega^I = -2\beta$. Then we have

$$d\omega^I = -2d\beta = -2\eta = -2i_\xi * \alpha.$$

Using the Einstein equations, we have

$$\alpha = Rc(\xi, \cdot) + \lambda^{-1} Rc(\xi, \xi)\xi^\flat. \tag{9.5}$$

Now, $i_\xi * \xi^\flat$ vanishes. Therefore, we see that $d\omega^I$ exactly cancels the failure for the twist one-form $\omega$ to be closed as in (9.1). So, $\omega + \omega^I$ is closed and we want to replace $\omega$ in the Geroch–Hansen formalism by $\omega + \omega^I$.

There seems to be a problem with the construction of $\omega^I$ above: there is gauge freedom in the potential $\beta$ for $\eta$. The one-form $\beta$ is allowed to change by a closed one-form. On $\mathbb{R}^3 \setminus \overline{B}^3$, a one-form is closed if and only if it is exact, so we can safely assume $\beta$ can only differ by an exact one-form. Mayerson [77, Section 3.3] provides a way to fix the gauge assuming the metric has the same form (5.12) as in the Thorne formalism. In that case, the twist one-form is still of the form of equation (6.6). The idea is to demand that to highest order, we do not change the twist one-form. In other words, we demand that $\omega_0^I = 0$ and

$$\omega_i^I = \sum_{l=1}^{\infty} \frac{S_{l-1}}{r^{l+1}}, \tag{9.6}$$

with the notation from the Thorne formalism. In that case, also the twist potential stays of the form (6.7). Mayerson [77, Section 3.3.2] proves that it is possible for $\omega^I$ to satisfy (9.6) by calculating the Ricci tensor in terms of spherical harmonics and subsequently expressing the condition on $\beta$ in terms of spherical harmonics.

**Definition 9.1.** Let $(M, g)$ be a stationary spacetime with Killing vector field $\xi$ such that $\eta$ defined by (9.2) and (9.3) is exact. Suppose the metric can be written in the form Equation 5.12 using harmonic coordinates that preserve the asymptotically flat form. Then the *improvement for the twist one-form* is a one-form $\omega^I$ defined by $d\omega^I = -2\eta$ and (9.6) and the *improved twist one-form* is $\omega + \omega^I$, where $\omega$ is the twist one-form (2.5).

Given the improved twist one-form, it is actually easy to define the multipole moments. We use exactly the same recursion relation as in the Geroch-Hansen formalism. Of course, we need some smoothness assumptions, but that is not so much different from vacuum. If the metric is given by (5.12) as in the Thorne formalism, also the proof of Theorem 6.4 carries over to prove that the multipole moments coincide when working in the gauge (9.6) [108]. Note that it is not clear whether we can always write the metric in the form of (5.12) in presence of matter.

We want to observe what $\omega^I$ is in the situations we have already studied: vacuum and electrovacuum. First, suppose the twist one-form is closed. That means $\xi^\flat \wedge Rc(\xi, \cdot) = 0$, but then we also have $\xi^\flat \wedge \alpha = 0$ by equation (9.5), from which we can conclude $\eta = 0$. But then we can also take $\beta = 0$ which is clearly of the form (9.6). Therefore, the improved twist one-form only differs from the original twist one-form if the latter fails to be closed. In other words, $\omega^I$ only appears if the twist one-form is not closed. In particular, we have $\omega^I = 0$ in vacuum, precisely as wanted. In electrovacuum, we have

$$\eta = -2d\psi \wedge d\varphi.$$

With $\omega^I$ from (7.5), we have

$$d\omega^I = 4d\psi \wedge d\varphi = -2\eta.$$

Therefore, we indeed improve the twist one-form in the same way as in Section 7.1, so the method is consistent with our gravitational multipole moments in electrovacuum.

There is a problem with the multipole moments in presence of matter. The method above only gives us gravitational multipole moments, and we cannot distinguish spacetimes using only the gravitational multipole moments. For example, the Kerr and Kerr-Newman spacetimes have the same gravitational multipole moments. Therefore, we also want to introduce so-called matter multipole moments. In vacuum, those matter multipole moments should all vanish and in electrovacuum we can simply use the electromagnetic multipole moments from Section 7.1. It is not clear how to define the matter multipole moments in general. One could hope to either expand the stress-energy tensor in spherical harmonics or one should extract suitable potentials from the stress-energy tensor.

## 9.2 Scalar field solutions

We want to have a look at multipole moments in some other specific class of exact solutions of the Einstein equations. In this section, we consider scalar field solutions. That is, we assume there is a (real or complex) scalar field $\phi$ on $M$ such that $(M, g)$ is a solution of the Einstein equations with stress-energy tensor

$$T = \frac{1}{2}\big(d\overline{\phi} \otimes d\phi + d\phi \otimes d\overline{\phi}\big) - \left(\frac{1}{2}|d\phi|_g^2 + V(\phi)\right)g, \tag{9.7}$$

where $V$ is a potential function. For a free scalar field of mass $m$, we must have $V(\phi) = \frac{1}{2}m^2|\phi|^2$. Note that there is also a complex conjugate contained in the term $|d\phi|_g^2$. For complex-valued functions $\phi$, we have

$$|d\phi|_g^2 = g^{\mu\nu}\partial_\mu\overline{\phi}\partial_\nu\phi.$$

Equation (9.7) also comes with a field equation for the scalar field. We can see it as if it is necessary for achieving the conservation law for the stress-energy tensor. If $\phi$ is complex-valued, it is best to see $V$ as a function of the real and imaginary parts of $\phi$, or alternatively as a function of $\phi$ and $\overline{\phi}$. In that case, we find

$$\Box_g\phi - 2\frac{\partial V}{\partial \overline{z}}(\phi) = 0. \tag{9.8}$$

With $V(\phi) = \frac{1}{2}m^2|\phi|^2$, this gives

$$\Box_g\phi - m^2\phi = 0,$$

which is precisely the Klein-Gordon equation with our sign convention. A scalar field solution of the Einstein equations is a spacetime $(M, g)$ together with a scalar field $\phi \in C^\infty(M)$ solving (9.8) for some potential $V$ and such that $(M, g)$ solves the Einstein equations with stress-energy tensor (9.7).

In the remainder of this section, we study two types of scalar field solutions. The first one are stationary scalar field solutions, where the scalar field must also be stationary. We drop stationarity of the scalar field in the last part of this section.

### Stationary scalar field solutions

**Definition 9.2.** A scalar field solution $(M, g)$ with scalar field $\phi$ is called *stationary* if $(M, g)$ is stationary with stationary vector field $\xi$ and $\mathcal{L}_\xi \phi = 0$.

Since $\mathcal{L}_\xi \phi = 0$, we also have $\mathcal{L}_\xi \overline{\phi} = 0$, which gives $\mathcal{L}_\xi T = 0$. That is what we need because the Lie derivatives of the Ricci tensor and the scalar curvature with respect to $\xi$ also vanish. Using the Einstein equations, we can express the Ricci tensor as

$$\begin{aligned} Rc = T - \frac{1}{2} g \operatorname{Tr}(T) &= \frac{1}{2}\big(d\overline{\phi} \otimes d\phi + d\phi \otimes d\overline{\phi}\big) - \Big(\frac{1}{2}|d\phi|_g^2 + V(\phi)\Big) g - \frac{1}{2}\Big(-|d\phi|_g^2 - 4V(\phi)\Big) g \\ &= \frac{1}{2}\big(d\overline{\phi} \otimes d\phi + d\phi \otimes d\overline{\phi}\big) + V(\phi) g. \end{aligned} \tag{9.9}$$

Therefore,

$$Rc(\xi, \cdot) = \frac{1}{2}\big(i_\xi\big(d\overline{\phi}\big) d\phi + i_\xi(d\phi) d\overline{\phi}\big) + V(\phi)\xi^\flat = V(\phi)\xi^\flat,$$

because $i_\xi d\phi = \mathcal{L}_\xi \phi = 0$ and similarly for $\overline{\phi}$. Therefore, $\xi^\flat \wedge Rc(\xi, \cdot) = 0$. By Theorem 2.12, this proves that the twist one-form defined by (2.5) is closed. Therefore, we do not need to improve the twist one-form in the sense of Definition 9.1.

Since the twist one-form is closed, the twist potential $f$ defined by $df = \omega$ exists. As usual, we assume that the first de Rham cohomology of the observer space $S$ vanishes. Therefore, we can still define the mass and angular momentum potentials in the Geroch–Hansen formalism according to Definition 4.1. We assume that $(M, g)$ is asymptotically flat in the sense of Definition 3.2 and we denote the one-point completion of $(S, h)$ by $\big(\widetilde{S}, \widetilde{h}\big)$ and we have the conformal factor $\Omega$ such that $\widetilde{h} = \Omega^2 h$. Then, like in Definition 4.9, we assume that the gravitational potentials $\widetilde{\phi}_A = \Omega^{-\frac{1}{2}} \phi_A$ extend to smooth functions on $\widetilde{S}$ for $A = M, J$.

With a stationary scalar field, it is also not difficult to think of matter multipole moments. We simply use $\phi$ as a potential. Since $\phi$ is stationary, i.e., $\mathcal{L}_\xi \phi = 0$, $\phi$ lives indeed on the observer space $S$. We assume that $\Omega^{-\frac{1}{2}} \phi$ extends to a smooth function on $\widetilde{S}$. This is not very unreasonable to expect. The asymptotic flatness condition ensures that the Ricci tensor, so also the stress-energy tensor, must fall off towards infinity [77]. Therefore, we expect that $\Omega^{-\frac{1}{2}} \phi$ is a well-defined function near $i^0$, where $\Omega$ is the conformal factor as in Definition 3.2. Of course, this is a technical assumption and it should be checked each time. Assuming $\Omega^{-\frac{1}{2}} \phi$ is smooth, we can simply use the function $\phi$ as a matter potential and apply Definition 4.5. If $\phi$ is a real scalar field, this gives one set of multipole moments describing the function. If $\phi$ is a complex scalar field, this gives two sets of multipole moments by taking the real and imaginary part, or one complex set of multipole moments.

**Definition 9.3.** Let $(M, g)$ be a stationary asymptotically flat scalar field solution of the Einstein equations with (real or complex) stationary scalar field $\phi \in C^\infty(M)$. Let $\Omega$ be the conformal factor in Definition 3.2 and assume that $\widetilde{\phi}_M = \Omega^{-\frac{1}{2}} \phi_M$, $\widetilde{\phi}_J = \Omega^{-\frac{1}{2}} \phi_J$ and $\widetilde{\phi} = \Omega^{-\frac{1}{2}} \phi$ extend to smooth functions on $\widetilde{S}$, where $\phi_M$ and $\phi_J$ are defined in Definition 4.1.

We end this section by looking at two examples of scalar field solutions of the Einstein equations. First, we consider the solutions discovered by Wyman [115]. They are a bit trivial, but that may not be bad to start with. They are static, spherically symmetric solutions to the Einstein equations in presence of a free, massless scalar field. The solutions found by Wyman are equivalent to the ones found by Janis, Newman and Winicour [58] as shown by Virbhadra [110]. The metric in coordinates $(t, r, \theta, \varphi)$ looks like

$$g = -\left(1 - \frac{2m}{\gamma r}\right)^{\gamma} dt^2 + \left(1 - \frac{2m}{\gamma r}\right)^{-\gamma} dr^2 + \left(1 - \frac{2m}{\gamma r}\right)^{1-\gamma} r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right),$$

for some positive constant $m$ and some constant $0 < \gamma \leq 1$. The resulting spacetime is an exact solution of the Einstein equations with scalar field

$$\phi = \sqrt{\frac{1-\gamma^2}{2}} \log\left(1 - \frac{2m}{\gamma r}\right). \tag{9.10}$$

If $\gamma = 1$, we are simply left with the Schwarzschild spacetime. Since the spacetime is static, the twist one-form vanishes. Moreover, there is no improved twist one-form needed because the scalar field is also independent of time. Looking at the Geroch-Hansen potentials, we can immediately conclude that $\phi_J = 0$. So we only have to calculate the mass and matter multipole moments. We have

$$\lambda = \left(1 - \frac{2m}{\gamma r}\right)^{\gamma}.$$

Moreover, the metric on the observer space is

$$h = dr^2 + \left(1 - \frac{2m}{\gamma r}\right) r^2 \left(d\theta^2 + \sin^2\theta d\varphi^2\right).$$

Note that this is the same as what we would get for the Schwarzschild metric if we replace $\frac{2m}{\gamma}$ by $2m$. Inspired by our work for the Kerr spacetime in Section 4.3, we introduce a new radial coordinate as

$$\overline{R} = \frac{2\gamma^2\left(r - \frac{m}{\gamma} - \sqrt{r\left(r - \frac{2m}{\gamma}\right)}\right)}{m^2}.$$

The coordinate transformation can be inverted using

$$r = \overline{R}^{-1}\left(1 + \frac{m}{\gamma}\overline{R} + \frac{m^2}{4\gamma^2}\overline{R}^2\right).$$

Then, it is easy to check that

$$\widetilde{h} = \Omega^2 h = d\overline{R}^2 + \overline{R}^2 d\theta^2 + \overline{R}^2 \sin^2\theta d\varphi^2,$$

with

$$\Omega = \frac{\overline{R}^2}{1 - \frac{m^2}{4\gamma^2}\overline{R}^2}.$$

We complete the observer space with $i^0$, which corresponds to $\overline{R} = 0$. Using Cartesian coordinates, we easily verify the conditions on the derivatives of $\Omega$, concluding that the Wyman solutions are asymptotically flat according to the definition by Geroch. For the mass potential, we have

$$\phi_M = \frac{1-\lambda^2}{4\lambda} = \frac{1}{4}\left(1 - \frac{2m}{\gamma r}\right)^{-\gamma} - \frac{1}{4}\left(1 - \frac{2m}{\gamma r}\right)^{\gamma} = \frac{1}{4}\left(\frac{1 - \frac{m}{2\gamma}\overline{R}}{1 + \frac{m}{2\gamma}\overline{R}}\right)^{-2\gamma} - \frac{1}{4}\left(\frac{1 - \frac{m}{2\gamma}\overline{R}}{1 + \frac{m}{2\gamma}\overline{R}}\right)^{2\gamma}.$$

Then,

$$\widetilde{\phi}_M = \Omega^{-\frac{1}{2}}\phi_M = \frac{1}{4}\overline{R}^{-1}\left(1-\frac{m}{2\gamma}\overline{R}\right)^{\frac{1}{2}}\left(1+\frac{m}{2\gamma}\overline{R}\right)^{\frac{1}{2}}\left(\left(\frac{1-\frac{m}{2\gamma}\overline{R}}{1+\frac{m}{2\gamma}\overline{R}}\right)^{-2\gamma}-\left(\frac{1-\frac{m}{2\gamma}\overline{R}}{1+\frac{m}{2\gamma}\overline{R}}\right)^{2\gamma}\right).$$

We can express $\widetilde{\phi}_M$ as a converging Taylor series around $\overline{R}=0$, and we see only even powers arise. Therefore, we can also express it as a converging power series in Cartesian coordinates, from which we achieve smoothness of $\widetilde{\phi}_M$ around $i^0$. In the limit with $\overline{R}\to 0$, we have

$$\widetilde{\phi}_M(i^0) = m,$$

so we recover $m$ as the mass of the system. The higher order mass multipole moments all vanish. In a Taylor expansion, $\widetilde{\phi}_M$ only contains (even) powers of $\overline{R}$ and no angular dependence. Therefore, all derivatives end up in the trace-part rather than the trace-free part, and after taking the trace-free part, all higher order multipole moments vanish. So, the gravitational multipole moments are the same as for the Schwarzschild spacetime.

We want to distinguish the Wyman solutions from the Schwarzschild spacetime using the matter multipole moments, so we should also calculate $\widetilde{\phi} = \Omega^{-\frac{1}{2}}\phi$. We have

$$\widetilde{\phi} = \Omega^{-\frac{1}{2}}\phi = \sqrt{\frac{1-\gamma^2}{2}}\overline{R}^{-1}\left(1-\frac{m^2}{4\gamma^2}\overline{R}^2\right)^{\frac{1}{2}}\log\left(\frac{1-\frac{m}{2\gamma}\overline{R}}{1+\frac{m}{2\gamma}\overline{R}}\right)^2.$$

Using a Taylor expansion, also here we see only even powers appear from which we can conclude that $\widetilde{\phi}$ is smooth around $i^0$. Moreover, in the limit $\overline{R}\to 0$ we get

$$\widetilde{\phi}(i^0) = -\frac{2m}{\gamma}\sqrt{\frac{1-\gamma^2}{2}}.$$

By exactly the same reasoning as for the mass multipole moments, no higher order matter multipole moments will appear. So, the only nonvanishing matter multipole moment is the monopole $\frac{-2m}{\gamma}\sqrt{\frac{1-\gamma^2}{2}}$. This is clearly nonzero when $\gamma \neq 1$, so it allows us to distinguishe the Wyman solution from the Schwarzschild spacetime. The behaviour of the multipole moments is similar to the Reissner-Nordström spacetime. In that case there are also only monopole moments and we can distinguish it from the Schwarzschild spacetime using the matter monopole moment.

The Wyman solution are, like Schwarzschild, a bit trivial when considering multipole moments. We expect more interesting behaviour when considering rotating scalar field solutions. For the rotating variant of the solutions above, the metric in Boyer-Lindquist coordinates is given by [47, 62]

$$\begin{aligned} g = &-\left(1-\frac{2mr}{\gamma\rho^2}\right)^{\gamma}dt^2 - 2a\sin^2\theta\left(1-\left(1-\frac{2mr}{\gamma\rho^2}\right)^{\gamma}\right)dtd\varphi + \left(1-\frac{2mr}{\gamma\rho^2}\right)^{1-\gamma}\rho^2\left(\frac{dr^2}{\Delta}+d\theta^2\right) \\ &+\left(-\left(1-\frac{2mr}{\gamma\rho^2}\right)^{\gamma}a^2\sin^2\theta + \left(1-\frac{2mr}{\gamma\rho^2}\right)^{1-\gamma}\rho^2 + 2a^2\sin^2\theta\right)\sin^2\theta d\varphi^2, \end{aligned} \tag{9.11}$$

with $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 + a^2 - \frac{2mr}{\gamma}$. If $\gamma = 1$, this boils down to the Kerr spacetime. This metric is a solution to the Einstein equations with the free, massless scalar field

$$\phi = \sqrt{\frac{1-\gamma^2}{2}}\log\left(1 - \frac{2mr}{\gamma\rho^2}\right).$$

Like above, we have

$$\lambda = \left(1 - \frac{2mr}{\gamma\rho^2}\right)^{\gamma},$$

and the metric on the observer space is the same as for Kerr but with $2m$ replaced by $\frac{2m}{\gamma}$. That is,

$$h = \frac{r^2 - \frac{2mr}{\gamma} + a^2\cos^2\theta}{r^2 - \frac{2mr}{\gamma} + a^2}dr^2 + \left(r^2 - \frac{2mr}{\gamma} + a^2\cos^2\theta\right)d\theta^2 + \left(r^2 - \frac{2mr}{\gamma} + a^2\right)\sin^2\theta d\varphi^2.$$

So, we easily see that the observer space is asymptotically flat by replacing $2m$ by $\frac{2m}{\gamma}$ everywhere. That is, we can take a new coordinate

$$\overline{R} = \frac{2\left(r - \frac{m}{2\gamma} - \sqrt{r^2 - \frac{2mr}{\gamma} + a^2}\right)}{\frac{m^2}{4\gamma^2} - a^2},$$

and

$$\Omega = \frac{\overline{R}^2}{\sqrt{\left(1 - \frac{1}{4}\left(\frac{m^2}{4\gamma^2} - a^2\right)\overline{R}^2\right)^2 - a^2\overline{R}^2\sin^2\theta}}.$$

Calculating the twist one-form (2.5) is more tedious than for the Kerr spacetime. Using Mathematica, we find

$$\omega = \frac{2a\cos\theta\left(\left(1 - \frac{2mr}{\gamma\rho^2}\right)\left(1 - \left(1 - \frac{2mr}{\gamma\rho^2}\right)^{\gamma}\right) + \frac{2ma^2r}{\rho^4}\sin^2\theta\right)}{\left(r^2 + a^2 - \frac{2mr}{\gamma}\right)\left(1 - \frac{2mr}{\gamma\rho^2}\right)^{1-\gamma}}dr + \frac{2am\left(r^2 - a^2\cos^2\theta\right)\sin\theta}{\rho^4\left(1 - \frac{2mr}{\gamma\rho^2}\right)^{1-\gamma}}d\theta. \tag{9.12}$$

Unfortunately, I did not manage to find a primitive for this one-form. Therefore, I have not been able to calculate the multipole moments.

### Travelling waves

Above, we assumed that the scalar field is stationary. This is similar to the situation in Chapter 7, where we assumed that the electromagnetic field is stationary. However, to define multipole moments it may not be necessary to assume this.

Suppose we have coordinates $\left(x^0 = t, x^1, x^2, x^3\right)$ such that $\frac{\partial}{\partial x^0}$ is a stationary vector field. Let $\phi$ be a function of the form

$$\phi(t, x^1, x^2, x^3) = e^{-ict}\psi(x^1, x^2, x^3), \tag{9.13}$$

for some complex function $\psi$ that is independent of $t$. Since (9.8) is a wave equation, it is not unreasonable to expect such solutions. For example, they play a very important role in electrodynamics where they describe electromagnetic waves as solutions of the flat wave equation [44, Chapter 9]. In these coordinates, it is easy to calculate

$$|d\phi|_g^2 = c^2 g^{00}|\psi|^2 + g^{\mu\nu}\partial_\mu\psi\partial_\nu\overline{\psi} + ig^{0j}c\big(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi}\big),$$

and

$$\begin{aligned}\frac{1}{2}\big(d\overline{\phi}\otimes d\phi + d\phi\otimes d\overline{\phi}\big) = c^2|\psi|^2dt^2 + \frac{1}{2}ic\big(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi}\big)\big(dt\otimes dx^j + dx^j\otimes dt\big)\\ + \frac{1}{2}\big(\partial_j\overline{\psi}\partial_k\psi + \partial_j\psi\partial_k\overline{\psi}\big)dx^j\otimes dx^k.\end{aligned} \tag{9.14}$$

If the potential $V$ only depends on $|\phi|$, it is easy to check using (9.7) that these relations tell us that the components of the stress energy tensor are independent of $t$. In other words, $\mathcal{L}_{\frac{\partial}{\partial x^0}}T = 0$. Therefore, it is not needed to assume that the scalar field is stationary to achieve that the stress-energy tensor is stationary.

Using (9.9) and (9.14), we see that the the $R_{0\mu}$-components of the Ricci tensor are given by

$$\begin{aligned}R_{00} &= c^2|\psi|^2 + V(|\psi|)g_{00},\\ R_{0j} &= \frac{1}{2}ic\big(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi}\big) + V(|\psi|)g_{0j}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(\frac{\partial}{\partial x^0}\right)^\flat \wedge Rc\left(\frac{\partial}{\partial x^0},\cdot\right) = \left(\frac{1}{2}icg_{00}\big(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi}\big) - g_{0j}c^2|\psi|^2\right)dt\wedge dx^j\\ + \frac{1}{2}icg_{0j}\big(\overline{\psi}\partial_k\psi - \psi\partial_k\overline{\psi}\big)dx^j\wedge dx^k,\end{aligned}$$

which is typically nonvanishing. Recall that the twist one-form is not closed in that case by Theorem 2.12, so we do need to improve the twist one-form in this case. With $\alpha$ as in (9.5), we have

$$\alpha = \alpha_\tau dx^\tau = \left(\frac{1}{2}ic\big(\overline{\psi}\partial_j\psi - \psi\partial_j\overline{\psi}\big) + c^2|\psi|^2\frac{g_{0j}}{g_{00}}\right)dx^j$$

In coordinates, applying the Hodge star operator gives

$$(*\alpha)_{\mu\nu\rho} = \varepsilon_{\sigma\mu\nu\rho}g^{\sigma\tau}\alpha_\tau = \varepsilon_{\sigma\mu\nu\rho}g^{\sigma j}\alpha_j,$$

so (9.3) yields

$$\eta_{\mu\nu} = \left(i_{\frac{\partial}{\partial x^0}} * \alpha\right)_{\mu\nu} = \varepsilon_{\sigma 0\mu\nu}g^{\sigma j}\alpha_j = -\varepsilon_{0i\mu\nu}g^{ij}\alpha_j.$$

To find the improved twist one-form, we need to ingrate $\eta$ and find $\beta$ such that $\eta = d\beta$. This cannot be done in general. It is impossible to continue the discussion. Even though this model is already quite old with research at the end of the sixties [64, 95], I lack of good explicit exact solutions. The ones I know are approximate solutions. If one has an exact solution, it would be interesting to see whether we can find $\beta$ and calculate multipole moments. We also need matter multipole moments. One could think of remembering $c$ and trying to apply Definition 4.5 to $\psi$.

# Chapter 10

# Conclusion and discussion

In this thesis, we have discussed multipole moments in stationary asymptotically flat spacetimes. In Part I, we have seen when a spacetime is stationary and asymptotically flat. For the former, we demand that a stationary vector field is complete, allowing us to work on a three-dimensional Riemannian manifold by Theorem 2.3 rather than the four-dimensional spacetime. However, the stationary vector field might not be complete. It would be interesting to investigate whether the observer space $S$ in Definition 2.2 is still ensured to be a manifold or whether there exists a counterexample where this is not the case. That would tell us whether we can maybe drop the completeness assumption.

For the geometric definitions of multipole moments, in vacuum or not, we use Geroch's definition of asymptotic flatness: Definition 3.2. It demands that we add a point "at infinity" to the observer space and the resulting space $\widetilde{S}$ must also be a three-dimensional Riemannian manifold, whose metric is conformal to the one on the observer space. To ensure uniqueness of the multipole moments, we also need that $\widetilde{S}$ together with its metric $\widetilde{h}$ is uniquely determined by the observer space $(S, h)$. The uniqueness result by Geroch [42, Appendix] was incorrect, but we can replace it by Theorem 3.3. One can also define asymptotic flatness in coordinates, but that yields an inequivalent definition as we discussed in Section 3.3. Unfortunately, the regularity class from Definition 3.8 is not sufficient to define multipole moments of order $l \geq 2$ using the Geroch–Hansen formalism. Ultimately, we want to know whether there are regularity classes that have a better resemblance with coordinate-based asymptotic flatness conditions and allow us to define multipole moments. Smoothness in Definition 3.2 is sometimes also a rather harsh assumption that may not be satisfied in cases where you want it to be satisfied [28, 42].

From Chapter 4 onwards, we have assumed that $H^1_{\mathrm{dR}}(S) = 0$ every now and then, where $S$ is the observer space. Recall that this is a reasonable assumption. If $(S, h)$ satisfies Definition 3.2, we can take a coordinate ball $B$ for $\left(\widetilde{S}, \widetilde{h}\right)$ centered at $i^0$. Since we are only interested in the local picture around $i^0$, we can restrict $S$ to $B \setminus \{i^0\}$. Correspondingly, we can also restrict $M$ to $\pi^{-1}(B \setminus \{i^0\})$ and $\widetilde{S}$ to $B$. Then $B \setminus \{i^0\}$, which is our new $S$, is diffeomorphic to $\mathbb{B}^3 \setminus \{0\}$. But the punctured ball is homotopy equivalent to the 2-sphere, so we find $H^1_{\mathrm{dR}}(S) = H^1_{\mathrm{dR}}(\mathbb{S}^2) = 0$. If one does not assume this, we have to assume that almost all closed one-forms we encountered are actually exact.

In this thesis, we introduced four geometric versions of multipole moments. First of all, we defined the Geroch–Hansen multipole moments in vacuum in Definition 4.9. The main issue that is left after assuming stationarity and asymptotic flatness is smoothness of the potentials on $\widetilde{S}$, which is partially solved by Lemma 4.4. In Definition 7.6, we generalised the Geroch–Hansen multipole moments to electrovacuum and in Definition 9.3, we generalised them to stationary scalar field solutions. Here, smoothness of the potentials is a bigger problem because we do not have a result like Lemma 4.4. Since Maxwell's equations constitute elliptic partial differential equations for the electric and magnetic scalar potentials and (9.8) is an elliptic partial differential equation for a stationary scalar field in a scalar field solutions, one might expect a similar result holds in these settings. The fourth geometric definition of multipole moments is discussed in Section 9.1 and is due to Mayerson [77]. It defines multipole moments in rather general spacetimes. The explicit conditions are mentioned in Definition 9.1, but it is difficult to digest what it means. It would be nice if this can be improved, maybe by fixing the gauge for $\omega^I$ in an alternative way.

For the Thorne multipole moments, we have defined asymptotic flatness in Definition 5.12. Basically, it says that if we have convergent series, then everything works. It would be good to know whether we can make this assumption more precise. Where the geometric multipole moments in vacuum, electrovacuum and scalar field solutions are treated in a mathematically rigorous way, this is a bit less so for the Thorne multipole moments in vacuum in Chapter 5 and in electrovacuum in Chapter 8. This should be better understood. The same is true when we assume both asymptotic flatness conditions are satisfied, which is what we do in Theorem 6.4. The result of Theorem 6.4 is very remarkable and powerful, it tells us that the Thorne and Geroch–Hansen multipole moments are equivalent. So, once we know one of them, we also know the other.

In Chapter 8, we developed a new method to find multipole moments in electrovacuum, similar to the Thorne formalism in vacuum. Even though we did not present a mathematically rigorous derivation, our physically intuitive argument shows that the construction most certainly works. It is also almost certain that the multipole moments are equivalent to the ones by Simon as we have discussed surrounding (8.16). It is left to go through the details.

In vacuum and electrovacuum, we have seen the multipole moments for the Kerr and Kerr–Newman spacetime, respectively, in both the geometric and coordinate-dependent formalism. In the geometric formalism, it is difficult to calculate multipole moments but we have seen it simplifies greatly in axisymmetric spacetimes in Section 4.3 and Section 7.2. We have been able to determine the multipole moments to arbitrary order. In the Thorne formalism, we cannot determine the multipole moments to arbitrary order, but with help of some software we can go to very high orders when working with the correct harmonic coordinates as in [59, 75]. When the spacetime is not axisymmetric, it is typically much easier to calculate the first few multipole moments using the Thorne formalisms than the Geroch–Hansen formalism.

In Section 9.2, we wanted to determine the multipole moments for the rotating JNW solution (9.11) [47, 62]. We have been able to show asymptotic flatness because the observer space is the same as for the Kerr spacetime but with a rescaled mass. Unfortunately, we have not been able to find a primitive function for the twist one-form (9.12). Once we have found it, we can find the multipole moments in the same way as for Kerr and Kerr–Newman using Theorem 4.17.

At the end of Section 9.2, we have briefly discussed scalar field solutions with time-dependent scalar field given by (9.13). Even though such models have been studied for a long time and are interesting in physics [64, 95], I am not aware exact, stationary, asymptotically flat examples. If there are any, it would be interesting to see whether we can determine the gravitational multipole moments using Mayerson's definition. We need the improved twist one-form because the original twist one-form is not closed. As far as I know, it could be the first solution where multipole moments can be calculated while there are time-dependent matter fields.

We end the conclusion and discussion with maybe the most important remark when we want to interpret measured multipole moments. Theorem 6.7 tells us that far away from the source, we can distinguish spacetimes based on their multipole moments. Assuming that we are in vacuum, this allows us to reconstruct the spacetime when measuring the multipole moments. However, the vacuum-assumption is key here. For example, the Kerr and Kerr–Newman solutions have the same gravitational multipole moments. In electrovacuum, we are still lucky because there is a similar result by Simon [102, Theorem 2]. However, there are also wilder examples of Newtonian objects that have the same multipole moments as Kerr [20]. This is also the reason why Mayerson's multipole moments are not as useful as the others: they do not provide a complete description in a certain setting. An ultimate goal would be to define "matter multipole moments" for any type of matter fields that can be measured. Then we can distinguish spacetimes irrespective of the type of matter. To be honest, I do not expect this is possible.

# Appendix A

# Tensor fields on the observer space

**Proposition 2.6.** *Let $(M, g)$ be a stationary spacetime with stationary vector field $\xi$ and with observer space $S$. There is a $C^\infty(S)$-module isomorphism between the set of tensor fields $T'$ on $S$ and the set of tensor fields $T$ on $M$ such that $\mathcal{L}_\xi T = 0$ and all possible contractions between $T$ and $\xi$ vanish. Moreover, the correspondence commutes with tensor products and contractions.*

*Proof.* The idea of the proof is that the pullback gives the one-to-one correspondence for covariant tensor fields. The assumption that the Lie derivative vanishes should be interpreted as if the tensor field is invariant under the action. The assumption that the contractions vanish should be interpreted as if the tensor field is horizontal to $S$. We prove the proposition in the following steps:

0. Define basic tensor fields and a $C^\infty(S)$-module structure on the space of basic tensor fields on $M$;
1. Construct the correspondence on covariant tensor fields;
2. Define a Riemannian metric on $S$ turning $\pi$ into a pseudo-Riemannian submersion;
3. Construct the correspondence for arbitrary $(k, l)$-tensor fields and observe it commutes with tensor products;
4. Characterise the correspondence for vector fields;
5. Show the correspondence commutes with contractions.

**Step 0: $C^\infty(S)$-module of basic tensor fields on $M$.** Let $\mathcal{T}^{(k,l)}(S)$ denote the $(k, l)$-tensor fields on $S$, and let $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$ be the $(k, l)$-tensor fields $T$ on $M$ satisfying $\mathcal{L}_\xi T = 0$ and such that all contractions of $T$ will $\xi$ or $\xi^\flat$ vanish. Then it is clear that $\mathcal{T}^{(k,l)}(S)$ is a $C^\infty(S)$-module, but it may not be immediately clear for $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$. Let $f'$ be a smooth function on $S$, then $f = \pi^* f' = f' \circ \pi$ is a smooth function on $M$. Moreover, for $p \in M$,

$$(\mathcal{L}_\xi f)(p) = \xi(\pi^* f')(p) = \xi_p(\pi^* f') = \left(f' \circ \pi \circ \theta^{(p)}\right)'(0).$$

But $\pi \circ \theta^{(p)}$ is the constant map that maps everything to the integral curve represented by $\theta^{(p)}$. Therefore, $f' \circ \pi \circ \theta^{(p)}$ is constant and $(\mathcal{L}_\xi f)(p) = 0$. Since $p \in M$ is arbitrary, $\mathcal{L}_\xi f = 0$.

Therefore, we can define scalar multiplication on $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$ by $f' \cdot T = fT = (\pi^* f')T$, which is again a basic tensor field. This turns $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$ into a $C^\infty(S)$-module by the properties of multiplying tensors with functions on $M$ and the fact that $\pi^*$ is linear and multiplicative on functions. To prove the result, we want to find isomorphisms $\Phi^{(k,l)} \colon \mathcal{T}^{(k,l)}(S) \to \mathcal{T}^{(k,l)}_{\text{bas}}(M)$.

**Step 1: correspondence for covariant tensor fields.** First, we consider covariant $k$-tensor fields. Let $A'$ be a covariant $k$-tensor field on $S$, then we have a covariant $k$-tensor field $A = \pi^* A'$ on $M$. Again,

$$(\mathcal{L}_\xi A)_p = \frac{d}{dt}\bigg|_{t=0} (\theta_t^* \pi^* A')_p = \frac{d}{dt}\bigg|_{t=0} ((\pi \circ \theta_t)^* A')_p = \frac{d}{dt}\bigg|_{t=0} (\pi^* A')_p = 0,$$

because $\pi(\theta(t,p)) = \pi(p)$ for all $p \in M$. This shows that $\mathcal{L}_\xi A = 0$. Since $d\pi_p(\xi_p) = 0$, it is also clear that contractions between $A$ and $\xi$ vanish. Hence, the pullback $\pi^*$ defines map from $\mathcal{T}^{(0,k)}(S)$ to $\mathcal{T}^{(0,k)}_{\text{bas}}(M)$, and we know it is a $C^\infty(S)$-module homomorphism that commutes with tensor products by the properties of tensor pullbacks. Since $\pi$ is a surjective smooth submersion, we see that $\pi^* \colon \mathcal{T}^{(0,k)}(S) \to \mathcal{T}^{(0,k)}_{\text{bas}}(M)$ is injective.

We want to take $\Phi^{(0,k)} = \pi^*$, but then the map must also be surjective. Let $A \in \mathcal{T}^{(0,k)}_{\text{bas}}(M)$ and take $\gamma \in S$, $v'_1, \ldots, v'_k \in T_\gamma S$ and $p \in M$ such that $\pi(p) = \gamma$. Since $d\pi_p \colon T_pM \to T_\gamma S$ is surjective, we can take $v_i \in T_pM$ such that $d\pi_p(v_i) = v'_i$, meaning $v_i$ is a lift of $v'_i$, for each $i$. Then we define a rough covariant $k$-tensor field $A'$ on $S$ by

$$A'_\gamma(v'_1, \ldots, v'_k) = A_p(v_1, \ldots, v_k).$$

We want to check that this is well-defined. That is, we want to show that the definition of $A'$ is independent of the choices we made. Let $w_i \in T_pM$ be another lift of $v'_i$, then $d\pi_p(v_i - w_i) = 0$, so $v_i - w_i \in \ker(d\pi_p)$. Therefore, there exists $c_i \in \mathbb{R}$ such that $v_i - w_i = c_i \xi_p$, for each $i$. But then we have

$$A_p(v_1, \ldots, v_k) = A_p(w_1 + c_1\xi_p, \ldots, w_k + c_k\xi_p) = A_p(w_1, \ldots, w_k),$$

because all possible contractions between $A$ and $\xi$ vanish. We also need to show $A'$ is independent of the choice of $p$. If we also have $\pi(q) = \gamma$, then $q = \theta(t,p)$ for some $t \in \mathbb{R}$. Moreover, we have $d(\theta_t)_p(v_i) \in T_qM$ such that $d\pi_q(d(\theta_t)_p(v_i)) = d(\pi \circ \theta_t)_p(v_i) = d\pi_p(v_i) = v'_i$. Therefore, $d(\theta_t)_p(v_i)$ is again a lift of $v'_i$, but now in $T_qM$. Since $\mathcal{L}_\xi A = 0$, we know $A$ is invariant under the flow of $\xi$, which gives

$$A_q(d(\theta_t)_p(v_1), \ldots, d(\theta_t)_p(v_k)) = (\theta_t^* A)_p(v_1, \ldots, v_k) = A_p(v_1, \ldots, v_k).$$

Therefore, $A'$ is independent of the choices. If $v_i$ is a lift of $v'_i$, then $av_i$ is a lift of $av'_i$, and if $w_i$ is also a lift of $w'_i$, then $v_i + w_i$ is a lift of $v'_i + w'_i$. Combined with the fact that $A_p$ is multilinear, this shows that $A'_\gamma$ is multilinear and $A'$ is a rough covariant $k$-tensor field on $S$. Hence, the construction of $A'$ is well-defined. We are left to prove that $A'$ is smooth. Let $U \subseteq S$ be a neighborhood of $\gamma$ such that we have a smooth section $\sigma \colon U \to M$ of $\pi \colon M \to S$. Since $A$ is a smooth covariant $k$-tensor field on $M$, this gives a smooth covariant $k$-tensor field $\sigma^* A$ on $U$. Now, $d\sigma_\gamma(v'_i)$ is a lift of $v'_i$ because $\sigma$ is a local section of $\pi$ and therefore,

$$A'_\gamma(v'_1, \ldots, v'_k) = A_{\sigma(\gamma)}(d\sigma_\gamma(v'_1), \ldots d\sigma_\gamma(v'_k)) = (\sigma^* A)_\gamma(v'_1, \ldots, v'_k).$$

So, $A'$ equals $\sigma^* A$ locally, and the latter is smooth. Since smoothness is a local property and the construction above can be done around any $\gamma \in S$, $A'$ is also smooth. Moreover, we easily see from the construction that $\pi^* A' = A$, so $\pi^* \colon \mathcal{T}^{(0,k)}(S) \to \mathcal{T}^{(0,k)}_{\text{bas}}(M)$ is surjective. Indeed, we can take $\Phi^{(0,k)} = \pi^*$ and it is a $C^\infty(S)$-module isomorphism that commutes with tensor products.

**Step 2: metric on $S$.** A particular covariant 2-tensor field on $M$ is the metric $g$. Let

$$h = g + \lambda^{-1} \xi^\flat \otimes \xi^\flat,$$

then it is a covariant 2-tensor field on $M$ such that $\mathcal{L}_\xi h = 0$ because $\mathcal{L}_\xi \xi = 0$ and $\mathcal{L}_\xi g = 0$ (note that $h$ differs by a factor $\lambda$ compared to equation (2.2)). Moreover, $h(\xi, \cdot) = 0 = h(\cdot, \xi)$, so the contractions of $h$ with $\xi$ vanish. By Step 1, there is a unique covariant 2-tensor field $h'$ on $S$ such that $\pi^* h' = h$. We will see that $h'$ is a Riemannian metric on $S$. Since $h$ is symmetric, we observe that $h'$ is also symmetric. Let $\gamma \in S$ and $v' \in T_\gamma S$, then we take $p \in M$ such that $\pi(p) = \gamma$ and we take $v \in T_p M$ to be the unique lift of $v'$ such that $g_p(\xi_p, v) = 0$. Then

$$h'_\gamma(v', v') = h_p(v, v) = g_p(v, v).$$

Since $g_p(\xi_p, v) = 0$, we know that $v$ is spacelike, so $g_p(v, v) \geq 0$ and $g_p(v, v) = 0$ if and only if $v = 0$. Therefore, $h'_\gamma(v', v') \geq 0$ with an equality if and only if $v' = d\pi_p(v) = 0$, so $h'$ indeed defines a Riemannian metric on $S$.

**Step 3: construction for arbitrary tensor fields.** A metric allows us to raise and lower indices of tensor fields. Given a $(k,l)$-tensor field $T'$ on $S$, we can lower all indices using $h'$ and we get a $(0, k+l)$-tensor field $T'^\flat$ on $S$. Then we apply $\pi^*$ so that we get $\pi^*\big(T'^\flat\big) \in \mathcal{T}^{(0,k+l)}_{\text{bas}}(M)$. Finally, we raise the lowered indices again using $g$ and we get a $(k,l)$-tensor field $\Phi^{(k,l)}(T') = \big(\pi^*\big(T'^\flat\big)\big)^\sharp$ on $M$. Lowering all indices using $h'$ gives a $C^\infty(S)$-module isomorphism from $\mathcal{T}^{(k,l)}(S)$ to $\mathcal{T}^{(k,l)}(S)$, and above we showed that $\pi^* \colon \mathcal{T}^{(0,k+l)}(S) \to \mathcal{T}^{(0,k+l)}_{\text{bas}}(M)$ is a $C^\infty(S)$-module isomorphism. The only thing that is left to show is that raising with $g$ gives a $C^\infty(S)$-module isomorphism between basic tensor fields on $M$. If so, we can take$\Phi^{(k,l)}$ to be the composition of these three $C^\infty(S)$-module isomorphisms, so it is one itself.

Suppose $A \in \mathcal{T}^{(0,k+l)}_{\text{bas}}(M)$ and define $T = A^\sharp$ by raising the first $k$ indices. Since $\xi$ is a Killing vector field, the Lie derivative $\mathcal{L}_\xi$ commutes with raising and lowering with $g$. Therefore, we also have $\mathcal{L}_\xi T = 0$. Moreover, contractions of $T$ with $\xi$ in one of the lower $l$ indices vanishes because it does for $A$. Contractions of $T$ with $\xi^\flat$ in one of the upper $k$ indices also vanishes because it is the same as lowering the index of $T$ back to as it was for $A$ and contracting with $\xi$. Hence, $T \in \mathcal{T}^{(k,l)}_{\text{bas}}(M)$. Moreover, raising gives a $C^\infty(M)$-module isomorphism, and restricts to a $C^\infty(S)$-module isomorphism between $\mathcal{T}^{(0,k+l)}_{\text{bas}}(M)$ and $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$.

This shows that $\Phi^{(k,l)}$ is a $C^\infty(S)$-module isomorphism between $\mathcal{T}^{(k,l)}(S)$ and $\mathcal{T}^{(k,l)}_{\text{bas}}(M)$. It is quite clear that the construction commutes with the tensor product. Suppose we have a tensor field $T' \otimes S'$ on $S$, then lowering all indices using $h'$ gives $(T' \otimes S')^\flat = T'^\flat \otimes S'^\flat$. The pullback on covariant tensor fields also commutes with the tensor product, and finally we raise all the indices again using $g$. Hence, we see that the isomorphism between tensor fields on $S$ and basic tensor fields on $M$ commutes with the tensor product.

From the construction, it is clear that $\Phi^{(k,l)}$ commutes with raising and lowering indices with $h'$ on $S$ and $g$ on $M$, respectively. In particular, we see that the inverse metric of $h'$

corresponds to raising $h$ with $g$, giving $g^{-1} + \lambda^{-1}\xi \otimes \xi$. In the construction of $\Phi^{(k,l)}$, we raised with respect to $g$. But this is the same as raising with $g^{-1} + \lambda^{-1}\xi \otimes \xi$ because $\pi^*\big(T'^{\flat}\big)$ the contraction of this covariant tensor field with $\xi$ in any index vanishes. Note that we can raise indices by any contravariant 2-tensor field, whether it is nondegenerate or not.

**Step 4: characterisation for vector fields.** We want to understand a bit better how the construction acts on contravariant tensor fields. Since we can write a contravariant tensor field as the sum of tensor products of vector fields and the construction commutes with sums and tensor products, it suffices to check what the map $\Phi^{(1,0)}$ looks like. Let $X \in \mathcal{T}^{(1,0)}_{\text{bas}}(M)$. For $p, q \in M$ such that $\pi(p) = \pi(q)$, there is a real number $t$ such that $q = t \cdot p$. From the fact that $\mathcal{L}_\xi X = 0$, we know that $X$ is invariant under the flow of $\xi$. But then

$$d\pi_{t\cdot p}(X_{t\cdot p}) = d\pi_{t\cdot p}(d(\theta_t)_p(X_p)) = d(\pi \circ \theta_t)_p(X_p) = d\pi_p(X_p).$$

Hence, there is a smooth vector field $X'$ on $S$ such that $d\pi_p(X_p) = X'_{\pi(p)}$ [72, Problem 8-18(c)]. We want to compare this to our construction, for which we use local coordinates. Let $p \in M$ and take coordinates $(t, x^1, x^2, x^3)$ on an open subset $U \subseteq M$ centered $p$ and $(y^1, y^2, y^3)$ on an open subset $V \subseteq S$ centered $\pi(p)$ such that $\pi(U) \subseteq V$, $\xi = \frac{\partial}{\partial t}$, and $\pi(t, x^1, x^2, x^3) = (y^1, y^2, y^3)$. Lowering the index of $X'$ gives $X'^{\flat} = h'_{ij}X'^j dy^i$. Then the pullback yields $\pi^*\Big(h'_{ij}X'^j\Big)dx^i$ and raising the index gives $g^{\mu i}\pi^*\Big(h'_{ij}X'^j\Big)\frac{\partial}{\partial x^\mu}$. By the construction, we have $\pi^*\Big(h'_{ij}\Big) = h_{ij}$ and $h_{0\mu} = 0$. Therefore,

$$g^{\mu i}\pi^*\big(h'_{ij}\big) = g^{\mu i}h_{ij} = g^{\mu\nu}h_{\nu j} = \delta^\mu_j + \lambda^{-1}\delta^\mu_0 g_{0j}.$$

So,

$$\Phi^{(1,0)}(X') = g^{\mu i}\pi^*\big(h'_{ij}X'^j\big)\frac{\partial}{\partial x^\mu} = \pi^*\big(X'^j\big)\frac{\partial}{\partial x^j} + \lambda^{-1}g_{0j}\pi^*\big(X'^j\big)\frac{\partial}{\partial t},$$

giving

$$\Big(\pi_*\Phi^{(1,0)}(X')\Big)_{\pi(p)} = d\pi_p\Big(\Phi^{(1,0)}(X')_p\Big) = \pi^*\big(X'^j\big)(p)\left.\frac{\partial}{\partial y^j}\right|_{\pi(p)} = X'^j(\pi(p))\left.\frac{\partial}{\partial y^j}\right|_{\pi(p)} = X'_{\pi(p)}.$$

Hence, $\big(\Phi^{(1,0)}\big)^{-1} = \pi_*$. This gives a definition of $\Phi^{(1,0)}$ that does not directly depend on the metric. The metric is still needed to define the basic vector fields and we need it for the inverse of $\pi_*$ to pick the vector field that is basic.

**Step 5: correspondence commutes with contractions.** We want to show that the construction commutes with contractions. To keep the notation simple, we only consider the trace of a $(1,1)$-tensor field and the argument can easily be generalised to contractions for $(k+1, l+1)$-tensor fields. Since a $(1,1)$-tensor field can be written as the sum of tensor products of vector fields with covector fields, we even restrict ourselves to such tensor fields. Let $\omega' \in \mathcal{T}^{(0,1)}(S) = \Omega^1(S)$, $X' \in \mathcal{T}^{(1,0)}(S) = \mathfrak{X}(S)$ and consider $\omega' \otimes X' \in \mathcal{T}^{(1,1)}(S)$. Then $\omega'(X')$ is the contraction of $\omega' \otimes X'$, and is a smooth function on $S$. Applying $\Phi^{(0,0)}$ to this function yields

$$f = \Phi^{(0,0)}(\omega'(X')) = \pi^*(\omega'(X')),$$

which is a smooth function on $M$ satisfying $\xi(f) = 0$. On the other hand, we can also bring $\omega' \otimes X'$ to $M$ and then take the trace. Let $\omega = \Phi^{(0,1)}(\omega')$ and $X = \Phi^{(1,0)}(X')$. Then $\omega = \pi^*\omega'$

and $X' = \pi_* X$, so $X'_{\pi(p)} = d\pi_p(X_p)$. Therefore, we have

$$\Phi^{(1,1)}(\omega' \otimes X') = \Phi^{(0,1)}(\omega') \otimes \Phi^{(1,0)}(X') = \omega \otimes X,$$

and the trace gives

$$\omega(X)(p) = (\pi^* \omega')_p(X_p) = \omega'_{\pi(p)}(d\pi_p(X_p)) = \omega'_{\pi(p)}(X'_{\pi(p)}) = f(p).$$

This concludes that contractions commute with bringing tensors on $S$ to basic tensors on $M$ and the reverse direction it is easily derived because the maps are isomorphisms. $\square$

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